GALOIS THEORY OF DIFFERENTIAL SCHEMES

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Abstract. Since 1883, Picard-Vessiot theory had been developed as the Galois theory of differential field extensions associated to linear differential equations. Inspired by categorical Galois theory of Janelidze, and by using novel methods of precategorical descent applied to algebraic-geometric situations, we develop a Galois theory that applies to morphisms of differential schemes, and vastly generalises the linear Picard-Vessiot theory, as well as the strongly normal theory of Kolchin.

Contents

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1. Introduction

1.1. **History.** Clasically, a Picard-Vessiot extension $(L, \delta_L)/(K, \delta_K)$ of differential fields yields a linear algebraic group

$$G = \mathrm{Gal}^{\mathrm{PV}}(L/K)$$

over the common field of constants $k = \mathrm{Const}(L, \delta_L) = \mathrm{Const}(K, \delta_K)$ such that there is a Galois correspondence between the intermediate differential field extensions and Zariski closed subgroups of G, and

$$G(k) \simeq \mathrm{Aut}((L, \delta_L)/(K, \delta_K)),$$

as explained in 5.1.

The Picard-Vessiot ring is a (K, δ_K)-algebra

$$(A, \delta_A)$$

with fraction field (L, δ_L), such that $(X, \delta_X) = \mathrm{Spec}(A, \delta_A)$ is a G-torsor over $(Y, \delta_Y) = \mathrm{Spec}(K, \delta_K)$ in the sense that

$$(X, \delta_X) \times_{(Y,\delta_Y)} (X, \delta_X) \simeq (X, \delta_X) \times_{(X_0,0)} (G, 0),$$

where we consider $X_0 = \mathrm{Spec}(k)$ and G as differential schemes endowed with zero derivations.

Janelidze realised that Picard-Vessiot theory fits into the framework of his categorical Galois theory through the adjunction

$$\begin{array}{c} \delta\text{-Aff} \\ S \Big\downarrow \dashv \Big\uparrow C \\ \mathrm{Aff} \end{array}$$

between the categories of affine differential schemes and affine schemes, where

$$S(\mathrm{Spec}(R,\delta_R)) = \mathrm{Spec}(\mathrm{Const}(R,\delta_R)), \qquad C(\mathrm{Spec}(R)) = \mathrm{Spec}(R,0).$$

He emphasised in [17] and [18] that the key object, the morphism of *relative Galois descent*, is in fact the torsor

$$f : (X,\delta_X) \to (Y,\delta_Y),$$

and not the extension of differential fields, and that the Picard-Vessiot Galois group agrees with the categorical Galois group,

$$\mathrm{Gal}^{\mathrm{PV}}(L/K) \simeq \mathrm{Gal}[f] = S(X \times_Y X),$$

as explained in 5.8.

During the visit of the second author to Tsukuba in 2023, Akira Masuoka asked whether categorical Galois theory approach automatically recovers the classical Picard-Vessiot Galois correspondence. To our surprise, we observed that Carboni-Janelidze-Magid correspondence from [10] in the affine context only establishes a correspondence between split affine quotients of X over Y and *effective* subgroups of $G = \mathrm{Gal}[f]$, i.e., those closed subgroups H such that G/H is represented by an affine scheme, see 5.10.

We realised that, contrary to the popular belief, Picard-Vessiot theory is not an entirely affine affair, and that the categorical Picard-Vessiot theory must be extended to the correspondence between the split *quasi-projective* fpqc quotients of (X,δ_X) over (Y,δ_Y), and closed subgroups of G, and that the intermediate differential fields from the classical correspondence appear as function fields of those quasi-projective quotients. Yves André also noted the failure of the affine Picard-Vessiot correspondence and fixed it in a different way by using *solution algebras* in [2].

On the other hand, the Galois theory of a *strongly normal* differential field extension $(L,\delta_L)/(K,\delta_K)$ of Kolchin [19] is known to give rise to a general algebraic group G over k and a general differential scheme (X,δ_X) that acts as a torsor for G through a relation analogous to the above, [6], [9], [31].

Michael Wibmer informed us of the idea of a number of researchers in differential algebra, including Jerry Kovacic, to interpret the strongly normal theory using categorical Galois theory, and this was independently posed as a desirable project by Janelidze in [18]. The difficulty lies in the fact that, with no reasonable notion of differential scheme, does the natural extension of the functor C above to a functor between schemes and differential schemes

$$\mathrm{Sch} \xrightarrow{\;C\;} \delta\text{-}\mathrm{Sch}$$

admit a left adjoint, so the classical categorical Galois theory cannot be invoked.

We resolve the apparent conundrum by constructing a very partial left adjoint using Bardavid's idea of *categorical scheme of leaves* [3], and by using an *indexed version* of categorical Galois theory from [8], eventually proving Theorem 7.2 that simultaneously explains the quasi-projective aspect of Picard-Vessiot theory and applies to strongly normal theory.

We proceed much further and develop a Galois theory of arbitrary differential scheme morphisms, as explained below. We hope that our work fulfils some of the wishes of Jerry Kovacic to bring the techniques of modern algebraic geometry to differential Galois theory.

1.2. **Differential algebraic geometry.** In this paper, a differential scheme

$$(X,\delta_X)=(X,(\mathscr{O}_X,\delta_X))$$

is a scheme $(X,\mathscr{O}_X)$ endowed with a derivation δ_X on its structure sheaf $\mathscr{O}_X$, i.e., with a vector field.

Given a scheme S, we define a category of S-differential schemes

$$\delta\text{-}\mathrm{Sch}_S$$

consisting of schemes over S endowed by S-derivations.

The spectrum of a differential ring has a natural structure of a differential scheme, affording a right adjoint

$$\mathrm{Spec}:\delta\text{-}\mathrm{Rng}^{\mathrm{op}}\to\delta\text{-}\mathrm{Sch}$$

to the global sections functor. A differential scheme is affine, if it is isomorphic to a spectrum of a differential ring, and the category of affine differential schemes is anti-equivalent to the category of differential rings, i.e.,

$$\delta\text{-}\mathrm{Aff}\simeq\delta\text{-}\mathrm{Rng}^{\mathrm{op}},$$

which is consistent with our discussion of affine differential schemes above.

A natural functor

$$C:\mathrm{Sch}\to\delta\text{-}\mathrm{Sch},\qquad (X,\mathscr{O}_X)\mapsto(X,(\mathscr{O}_X,0))$$

allows us to consider a scheme as a differential scheme with a trivial derivation/zero vector field, but it does not have a left adjoint. Based on the ideas of Bardavid bardavid [3, 4.2], we define a *categorical scheme of leaves*

$$\pi_0(X)$$

of a differential scheme (X,δ_X) to be the morphism

$$(X,\delta_X)\to C(\pi_0(X))$$

which is universal from (X,δ_X) to C. It exists only for rare differential schemes (X,δ_X), but this construction provides a partial left adjoint π_0 to C when it does.

1.3. **Differential schemes as precategory actions and descent.** We make a key novel observation that S-differential schemes can be viewed as actions of an internal precategory

$$\mathbb{D}(S)$$

in $\mathrm{Sch}_{/S}$ associated to infinitesimal augmentations of S that we can symbolically write

$$S[\epsilon_0,\epsilon_1]/(\epsilon_0^2,\epsilon_0\epsilon_1,\epsilon_1^2)\mathrel{\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}} S[\epsilon]/(\epsilon^2)\mathrel{\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}} S$$

i.e., that there is an equivalence of categories (4.5)

$$\delta\text{-}\mathrm{Sch}_S\simeq(\mathrm{Sch}_{/S})^{\mathbb{D}(S)}.$$

It was previously known that differential schemes can be viewed as 'actions of a pointed set' associated to the augmentation structure $S[\epsilon^2]/(\epsilon^2)\to S$, but the observation that we can view them as precategory actions gives us a very direct access to a theory of descent, given that precategory actions are essentially generalised descent data. More generally, we can translate problems from differential algebraic geometry into the known realm of algebraic geometry.

In algebraic geometry, descent usually works only for very specific indexed data on schemes, so we work with a chosen pseudofunctor

$$\mathscr{P} : (\mathrm{Sch}_{/S})^{\mathrm{op}} \to \mathbf{Cat},$$

and extend it in a natural way, using precategory actions, to a pseudofunctor on differential schemes

$$\delta\text{-}\mathscr{P} : \delta\text{-}\mathrm{Sch}_S^{\mathrm{op}} \to \mathbf{Cat}.$$

In the special case when $\mathscr{P}(V)$ is a certain class of scheme morphisms with target V, the category $\delta\text{-}\mathscr{P}(X, \delta_X)$ consists of differential scheme morphisms with target (X, δ_X) whose underlying scheme morphism belongs to the class $\mathscr{P}$.

If $\mathbb{X} = (X_2, X_1, X_0)$ is a precategory in $\mathrm{Sch}_{/S}$ that corresponds to a differential scheme (X, δ_0) viewed as an $\mathbb{D}(S)$-action, then the categorical scheme of leaves of (X, δ_X) agrees with the scheme of connected components of $\mathbb{X}$,

$$\pi_0(X) \simeq \pi_0(\mathbb{X}) = \mathrm{Coeq}\,(X_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} X_0)$$

whenever either of them exist in $\mathrm{Sch}_{/S}$.

Moreover, (X, δ_X) is *simple* with respect to $\mathscr{P}$, provided the above coequaliser exists and is universal for $\mathscr{P}$. This condition ensures that the precategory morphism

$$\eta_X : \mathbb{X} \to \mathbf{\Delta}(\pi_0(\mathbb{X}))$$

to the dicrete precategory associated to the scheme $\pi_0(X)$ is of *precategorical descent*, i.e., the induced functor of precategory actions

$$C_X : \mathscr{P}(\pi_0(X)) \simeq \mathscr{P}^{\mathbf{\Delta}(\pi_0(\mathbb{X}))} \xrightarrow{\eta_{\mathbb{X}}^*} \mathscr{P}^{\mathbb{X}} \simeq \delta\text{-}\mathscr{P}(X, \delta_X)$$

is fully faithful.

By developing the theory of descent in the category of differential schemes as precategory actions, we establish in 4.49 that a morphism

$$f : (X, \delta_X) \to (Y, \delta_Y)$$

is of effective descent for $\delta\text{-}\mathscr{P}$, if the associated morphism (f_0, f_1, f_2) of precategory actions satisfies that the underlying scheme morphism $f_0 : X \to Y$ is of effective descent for $\mathscr{P}$, and f_1 is descent for $\mathscr{P}$.

Consequently, such an f is of effective descent for the fibration of polarised quasi-projective differential morphisms if f_0 is faithfully flat quasi-compact (fpqc). Moreover, if the target of f is a spectrum of a differential field, then it is of effective descent for the class of quasi-projective differential morphisms.

1.4. **Galois theory of differential schemes.** A good understanding of the two types of descent provides all the ingredients needed for a Picard-Vessiot style categorical Galois theory of differential schemes. The framework for the indexed version of categorical Galois theory involves the following choices:

(1) let S be a scheme, and

$$\mathscr{X} = \mathrm{Sch}_{/S}$$

be the category of schemes over S;

(2) let $\mathscr{A}$ be the full subcategory of δ-Sch_S of those S-differential schemes that have a categorical scheme of leaves, hence we obtain a functor
$$\pi_0 : \mathscr{A} \to \mathscr{X};$$
(3) a pseudo-functor
$$\mathscr{P} : \mathscr{X}^{\mathrm{op}} \to \mathbf{Cat},$$
determining a pseudofunctor δ-$\mathscr{P} : \mathscr{A}^{\mathrm{op}} \to \mathbf{Cat}$;

(4) let
$$C = C^{\mathscr{P}} : \mathscr{P} \circ \pi_0 \to \delta\text{-}\mathscr{P}$$
be the pseudo-natural transformation whose component at $(X, \delta) \in \mathscr{A}$ is the functor C_X discussed above.

We define the category
$$\mathrm{Split}_C(f)$$
as the full subcategory of objects $P \in \delta$-$\mathscr{P}(Y)$ such that $f^*P \simeq C_X(Q)$ for some $Q \in \mathscr{P}(\pi_0(X))$.

We define a morphism $f : (X, \delta_X) \to (Y, \delta_Y)$ in $\mathscr{A}$ to be *pre-Picard-Vessiot* with respect to $\mathscr{P}$, if

cls-desc (i) f is a morphism of effective descent for δ-$\mathscr{P}$;

cube-simple (ii) X, $X \times_Y X$, $X \times_Y X \times_Y X$ are simple for $\mathscr{P}$.

If we have

(5) a suitable (in the sense of 6.2) pseudo-natural transformation
$$U : \mathscr{P} \Rightarrow \mathscr{S}$$
to a fibration $\mathscr{S}$ associated with a class of scheme morphisms,

then we define f to be *Picard-Vessiot* with respect to U if, in addition to the first, it satisfies the strengthening of the second condition,

(ii') X is simple for $\mathscr{S}$ and f is auto-split for $\mathscr{S}$ in the sense that $f \in \mathrm{Split}_{C^{\mathscr{S}}}(f)$.

The assumption that f is pre-Picard-Vessiot for $\mathscr{P}$ ensures that the kernel-pair groupoid
$$\mathbb{G}_f : \qquad X \times_Y X \times_Y X \rightrightarrows X \times_Y X \rightleftarrows X$$
is a category in $\mathscr{A}$, and that we can define the *Galois precategory*
$$\mathrm{Gal}[f] = \pi_0(\mathbb{G}_f)$$
as a precategory in $\mathscr{X}$. Moreover, if f is Picard-Vessiot for U, then it is also pre-Picard-Vessiot for $\mathscr{P}$ and $G[f]$ happens to be an internal *groupoid* in $\mathscr{X}$.

Theorem (Galois theorem for differential schemes, 6.11)**.** *A pre-Picard-Vessiot morphism f for $\mathscr{P}$ induces an equivalence*
$$\mathrm{Split}_C(f) \simeq \mathscr{P}^{\mathrm{Gal}[f]}$$
between the category of objects of δ-$\mathscr{P}(Y)$ that are C-split by f and the category of $\mathscr{P}$-actions of the precategory $\mathrm{Gal}[f]$.

If f is Picard-Vessiot for U, the latter becomes the category of $\mathscr{P}$-actions of the groupoid $\mathrm{Gal}[f]$.

From our perspective,

Differential Galois theory = (precategorical) descent + categorical Galois theory.

More precisely, condition (i) from the definition of pre-Picard-Vessiot morphisms ensures effective classical descent for precategory actions, while condition (ii) ensures the new type of precategorical descent where needed to apply the powerful abstract techniques of categorical Galois theory.

Our Galois precategory appears to be a differential scheme version of truncated to level 2 higher differential Galois groups developed by Ayoub in ayoub [?] as foliated homotopy types of a differential field. We hope to be able to pursue our techniques in the direction of higher category theory along similar lines.

1.5. **Applications.** Using the above template Galois theorem, we prove the following result, which simultaneously refines classical Picard-Vessiot theory and strongly normal theory of differential field extensions.

Corollary (Quasi-projective differential Galois correspondence 7.3)**.** *Let* (K, δ) *be a differential field of characteristic* 0 *with the field of constants* k*, let* $S = \mathrm{Spec}(k)$ *and assume*

$$f : (X, \delta) \to (Y, \delta) = \mathrm{Spec}(K, \delta)$$

is a quasi-projective morphism of S*-differential schemes such that* X *is integral and its only leaf is its generic point and* f *is auto-split. Then there is a Galois correspondence between split* S*-differential quasi-projective fpqc quotients of* f *in* $\mathscr{A}$ *and closed subgroups of the* S*-algebraic group* $\mathrm{Gal}[f]$*, which takes*

$$\begin{array}{ccc} & & X \\ & \swarrow & \Big\downarrow f \\ P & & \\ & {\scriptstyle p}\searrow & \\ & & Y \end{array}$$

to

$$\mathrm{Gal}[X \to P].$$

Conversely, a closed subgroup G' *of* $\mathrm{Gal}[f]$ *corresponds to the quotient*

$$X/G'.$$

As far as we are aware, we are able to formulate the first result in differential Galois theory which works over differential schemes with arbitrary categorical scheme of constants.

Theorem (Polarised quasi-projective differential Galois theory, 7.7)**.** *Let* $f : (X, \delta_X) \to (Y, \delta_Y)$ *be a morphism of* S*-differential schemes such that*

(1) *the underlying scheme morphism* $X \to Y$ *is fpqc;*
(2) (X, δ_X) *is simple with respect to* S*-scheme morphisms, with scheme of leaves* G_0*;*
(3) *there is an* S*-morphism* $G_1 \to G_0$ *such that*

$$(X, \delta_X) \times_{(Y, \delta_Y)} (X, \delta_X) \simeq (X, \delta_X) \times_{(G_0, 0)} (G_1, 0).$$

Then f is Picard-Vessiot for the forgetful functor from polarised morphisms to scheme morphisms, $\mathrm{Gal}[f]$ *is the groupoid* $(G_1 \rightrightarrows G_0)$ *and we have an equivalence between the category of quasi-projective polarised S-differential morphisms* $(P, \mathscr{L}_P) \to Y$ *split by f and the category of quasi-projective polarised actions* $(Q, \mathscr{L}_Q) \to G_0$ *of* $\mathrm{Gal}[f]$.

Parametric differential equations. The specialisation of Galois precategory result 6.13 makes our Galois theory suitable for treating parametric systems of differential equations where the parameters come from a constant scheme. We illustrate this in 7.8 on an example of an S-differential family $f : X \to Y$ of elliptic curves endowed with a vector field in such a way that $\pi_0(X) = \pi_0(Y) = S$ and the Galois groupoid is a family of elliptic curves over S

$$\mathrm{Gal}[f] = E \rightrightarrows S.$$

As the parameter s varies over $S(L)$, the specialisation formula 6.13 give that

$$\mathrm{Gal}[f]_s = \mathrm{Gal}[f_s] = E_s,$$

as an algebraic group over L, so our Galois groupoid specialises to classical Galois groups of strongly normal extensions associated with f_s calculated by Kolchin in kolchin-sn [19].

The Galois groupoid of a differential equation. In 7.9, we show that our notion of a Picard-Vessiot morphism can be used to study symmetries of linear differential equations through a canonical Galois groupoid, without making a non-canonical choice of a Picard-Vessiot extension as in the classical theory. We study a Picard-Vessiot morphism of differential schemes $f : X \to Y$ where X is associated to the full universal solution algebra of the Airy equation

$$y'' = xy,$$

which yields a Galois groupoid of the form

$$\mathrm{Gal}[f] = G_1 \rightrightarrows G_0,$$

where the points of the object of objects G_0 correspond to choices of Picard-Vessiot extensions, and the object of morphisms G_1 encodes isomorphisms between different choices of Picard-Vession extensions.

This is scheme-theoretic generalisation of Deligne's notion of the fundamental groupoid of a Tannakian category where the object of object consists of fibre functors, and the object of morphisms consists of isomorphisms between fibre functors. Indeed, given the Tannakian category of finite-dimensional vector spaces with a connection over a differential field, fibre functors correspond to Picard-Vessiot extensions, and we obtain a perfect analogy with our groupoid deligne-tannakien [11].

1.6. **Layout of the paper.** A reader interested only in differential Galois theory can start perusing the paper from Section 4, and occasionally look up the prerequisites from previous sections. There, we develop differential algebraic geometry, and discuss the numerous benefits of our original approach to differential schemes as precategory actions, including the consideration of the categorical schemes of leaves as connected components of precategories, simplicity of differential schemes through universality of connected components, polarised differential schemes. One of the most important topics we develop is the theory of descent for differential scheme morphisms.

In Section 5, we explore the extent to which the classical categorical Galois theory of Janelidze explains the affine Picard-Vessiot Galois correspondence.

In Section 6, we reap the benefits of the flexibility that the indexed version of categorical Galois theory brings to differential algebraic geometry, and formulate a very general template for the Galois theory of differential schemes. We apply this theory in concrete settings of quasi-projective differential Galois correspondence that works over a base differential field, and polarised quasi-projective differential Galois theory that works over arbitrary differential schemes.

On the other hand, we must emphasise that previous sections contain a myriad of results that may be of independent interest in category theory, especially in view of connections to recent work related to descent theory [22], [23], [29], [24] that we do not fully understand yet.

In Section 2, we discuss precategories and their actions, which, viewed as generalised descent data, give rise to a new form of *precategorical descent,* and find sufficient conditions for effective descent. We develop a whole calculus of pullbacks of descent data, and apply it to obtain a result on descent of quasi-projective morphisms. We also discuss classical descent of precategory actions that gets applied in the differential context later.

In Section 3, we recall the foundations of Janelidze's categorical Galois theory, its indexed form from his book with Borceux, and we expand the minute details of the Carboni-Magid-Janelidze categorical Galois correspondence that are implicit in the original paper [10].

This paper was largely motivated by numerous discussions with George Janelidze, Andy Magid, Akira Masuoka, Tom Scanlon and Michael Wibmer, so we thank them all for sharing their time and knowledge with us.

2. Descent

s:descent

Throughout this appendix, let $\mathscr{C}$ denote a category with pullbacks, and let

$$\mathscr{P} \to \mathscr{C}$$

be a fibred category equipped with a cleavage. Equivalently, we have an indexed category associated to a pseudofunctor

$$P : \mathscr{C}^{\mathrm{op}} \to \mathbf{Cat}$$

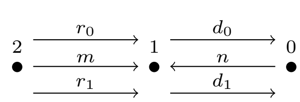
where, for an object U in $\mathscr{C}$, the fibre of $\mathscr{P}$ over U is the U-indexed component of P,

$$\mathscr{P}(U) = P(U),$$

and, for a morphism $U \xrightarrow{f} V$ in $\mathscr{C}$, we have a pullback functor

$$f^* = P(f) : \mathscr{P}(V) \to \mathscr{P}(U).$$

2.1. **Precategories and their actions.** Let Δ_2 be the diagram category



with

$$\begin{aligned} d_0r_1 &= d_1r_0, & d_0n &= \mathrm{id}_0,\\ d_0m &= d_0r_0, & d_1n &= \mathrm{id}_0,\\ d_1m &= d_1r_1. \end{aligned}$$

A *precategory* in a category $\mathscr{C}$ is a functor

$$\mathbb{C} : \Delta_2 \to \mathscr{C}.$$

Equivalently, it is a diagram

$$C_2 \mathrel{\substack{\xrightarrow{r_0}\\ \xrightarrow{m}\\ \xrightarrow{r_1}}} C_1 \mathrel{\substack{\xrightarrow{d_0}\\ \xleftarrow{n}\\ \xrightarrow{d_1}}} C_0$$

in $\mathscr{C}$, where the morphisms satisfy the relations indicated above.

The category of precategories in $\mathscr{C}$ is the functor category

$$\mathbf{PreCat}(\mathscr{C}) = [\Delta_2, \mathscr{C}].$$

precat-discr

Definition 2.2. The discrete precategory associated to an object U of $\mathscr{C}$ is the diagram

$$\boldsymbol{\Delta}(U): \qquad U \mathrel{\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}} U \mathrel{\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}} U$$

where all the morphisms are identities on U.

precat-ker-pair

Definition 2.3. The precategory associated to the kernel pair of a morphism $u : U' \to U$ in a category $\mathscr{C}$ admitting pullbacks is the diagram

$$\mathbb{G}_u : \qquad U' \times_U U' \times_U U' \mathrel{\substack{\xrightarrow{\pi_{01}}\\ \xrightarrow{\pi_{02}}\\ \xrightarrow{\pi_{12}}}} U' \times_U U' \mathrel{\substack{\xrightarrow{\pi_0}\\ \xleftarrow{\Delta}\\ \xrightarrow{\pi_1}}} U'$$

which happens to be a groupoid without the inversion of morphisms named. Note that, using the notation from 2.2,

$$\boldsymbol{\Delta}(U) = \mathbb{G}_{\mathrm{id}_U}.$$

Definition 2.4. For $\mathbb{C} \in \mathbf{PreCat}(\mathscr{C})$, the category of $\mathbb{C}$-*actions in* $\mathscr{P}$ is the bilimit

$$\mathscr{P}^{\mathbb{C}} = \lim(\mathscr{P} \circ \mathbb{C})$$

of the diagram of categories and functors

$$\mathscr{P}(C_2) \mathrel{\substack{\xleftarrow{r_0^*}\\ \xleftarrow{m^*}\\ \xleftarrow{r_1^*}}} \mathscr{P}(C_1) \mathrel{\substack{\xleftarrow{d_0^*}\\ \xrightarrow{n^*}\\ \xleftarrow{d_1^*}}} \mathscr{P}(C_0).$$

More explicitly, an action $P \in \mathscr{P}^{\mathbb{C}}$ consists of objects

$$P_2 \in \mathscr{P}(C_2), \qquad P_1 \in \mathscr{P}(C_1), \qquad P_0 \in \mathscr{P}(C_0),$$

and isomorphisms

$$
\begin{array}{lll}
P_2 \xrightarrow{\rho_0} r_0^* P_1, & P_1 \xrightarrow{\delta_0} d_0^* P_0, & P_0 \xrightarrow{\eta} n^* P_1, \\
P_2 \xrightarrow{\mu} m^* P_1, & P_1 \xrightarrow{\delta_1} d_1^* P_0, & \\
P_2 \xrightarrow{\rho_1} r_1^* P_1, & &
\end{array}
$$

such that the diagram

$$
\begin{tikzcd}
n^* d_0^* P_0 & n^* P_1 \arrow[l, "n^*\delta_0"'] \arrow[r, "n^*\delta_1"] & n^* d_1^* P_0 \\
& P_0 \arrow[ul, "\sim"] \arrow[u, "\eta"'] \arrow[ur, "\sim"'] &
\end{tikzcd}
$$

commutes in $\mathscr{P}(C_0)$, and the diagram

$$
\begin{tikzcd}
& m^* d_0^* P_0 & m^* P_1 \arrow[l, "m^*\delta_0"'] \arrow[r, "m^*\delta_1"] & m^* d_1^* P_0 \arrow[dr, "\sim"] & \\
r_0^* d_0^* P_0 \arrow[ur, "\sim"] & & P_2 \arrow[u, "\mu"'] \arrow[dl, "\rho_0"'] \arrow[dr, "\rho_1"] & & r_1^* d_1^* P_0 \\
& r_0^* P_1 \arrow[ul, "r_0^*\delta_0"] \arrow[d, "r_0^*\delta_1"'] & & r_1^* P_1 \arrow[ur, "r_1^*\delta_1"'] \arrow[d, "r_1^*\delta_0"] & \\
& r_0^* d_1^* P_0 \arrow[rr, "\sim"] & & r_1^* d_0^* P_0 &
\end{tikzcd}
$$

commutes in $\mathscr{P}(C_2)$, where the unnamed arrows are coherence isomorphisms.

actions-self-ind **Definition 2.5.** The category of $\mathbb{C}$-*actions in* $\mathscr{C}$ is

$$\mathscr{C}^{\mathbb{C}} = \mathrm{Self}(\mathscr{C})^{\mathbb{C}},$$

where $\mathrm{Self}(\mathscr{C})(C) = \mathscr{C}_{/C}$ is the self-indexing of $\mathscr{C}$ by slicing.

gen-dd *Remark* 2.6. By considering the composite

$$
\begin{tikzcd}
d_0^* P_0 & P_1 \arrow[l, "\delta_0"'] \arrow[r, "\delta_1"] & d_1^* P_0
\arrow[from=1-1, to=1-3, bend left, "\varphi"]
\end{tikzcd}
$$

we see that an action P is equivalently given by an object $P_0 \in \mathscr{P}(C_0)$ and an isomorphism $\varphi : d_0^* P_0 \to d_1^* P_0$ in $\mathscr{P}(C_1)$ such that the diagram

$$
\begin{tikzcd}
n^* d_0^* P_0 \arrow[rr, "n^*\varphi"] & & n^* d_1^* P_0 \\
& P_0 \arrow[ul, "\sim"] \arrow[ur, "\sim"'] &
\end{tikzcd}
$$

commutes in $\mathscr{P}(C_0)$, and the *cocycle condition* holds, i.e., the diagram

$$\begin{array}{ccccc} & m^*d_0^*P_0 & \xrightarrow{m^*\varphi} & m^*d_1^*P_0 & \\ {\scriptstyle\sim}\nearrow & & & & \searrow{\scriptstyle\sim} \\ r_0^*d_0^*P_0 & & & & r_1^*d_1^*P_0 \\ {\scriptstyle r_0^*\varphi}\searrow & & & & \nearrow{\scriptstyle r_1^*\varphi} \\ & r_0^*d_1^*P_0 & \xrightarrow{\sim} & r_1^*d_0^*P_0 & \end{array}$$

commutes in $\mathscr{P}(C_2)$.

A morphism

$$(P,\varphi) \to (P',\varphi')$$

between two $\mathbb{C}$-actions considered this way is given by a morphism $\psi : P_0 \to P_0'$ in $\mathscr{P}(C_0)$ such that the diagram

$$\begin{array}{ccc} d_0^*P & \xrightarrow{d_0^*\psi} & d_0^*P' \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi'} \\ d_1^*P & \xrightarrow{d_1^*\psi} & d_1^*P' \end{array}$$

commutes in $\mathscr{P}(C_1)$.

`action-fibred` *Remark* 2.7. In terms of the fibration $\mathscr{P} \to \mathscr{C}$, an action in $\mathscr{P}^{\mathbb{C}}$ is given by a precategory $\mathbb{P} \in \mathbf{PreCat}(\mathscr{P})$

$$P_2 \; \begin{array}{c} \xrightarrow{r_0} \\ \xrightarrow{m} \\ \xrightarrow{r_1} \end{array} \; P_1 \; \begin{array}{c} \xrightarrow{d_0} \\ \xleftarrow{n} \\ \xrightarrow{d_1} \end{array} \; P_0$$

living above $\mathbb{C}$, where the arrows in $\mathbb{P}$ are cartesian.

2.8. **Connected components of precategories.**

`def-conn-comp` **Definition 2.9.** The *object of connected components* of a precategory $\mathbb{C} \in \mathbf{PreCat}(\mathscr{C})$ is the reflexive coequaliser

$$C_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} C_0 \longrightarrow \pi_0(\mathbb{C}),$$

provided it exists in $\mathscr{C}$.

If $\mathscr{P} : \mathscr{C}^{\mathrm{op}} \to \mathbf{Cat}$ is a pseudofunctor, the object of connected components of an action $P \in \mathscr{P}^{\mathbb{C}}$ is defined as the object of connected components of the precategory $\mathbb{P} \in \mathbf{PreCat}(\mathscr{P})$ associated to $\mathscr{P}$ via 2.7,

$$\pi_0(P) = \pi_0(\mathbb{P}),$$

provided it exists in the fibred category $\mathscr{P}$ over $\mathscr{C}$.

`coeq-above-coeq` **Lemma 2.10.** *With the notation from 2.9, suppose that $\pi_0(\mathbb{C})$ and $\pi_0(P)$ exist. Then $\pi_0(P)$ projects to $\pi_0(\mathbb{C})$, i.e., a reflexive coequaliser*

$$P_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} P_0 \xrightarrow{\rho} \pi_0(P)$$

in $\mathscr{P}$ projects to a reflexive coequaliser

$$C_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} C_0 \xrightarrow{c} \pi_0(\mathbb{C})$$

in $\mathscr{C}$.

Proof. Writing $p : \mathscr{P} \to \mathscr{C}$ for the given fibration, and using the fact that c is a coequaliser, there exists a unique morphism $t : S \to p(\pi_0(P))$ in $\mathscr{C}$ such that $p(\rho) = tc$. Writing $\theta : t^*\pi_0(P) \to \pi_0(P)$ for a cartesian lift of t, there is a unique $\gamma : P_0 \to t^*\pi_0(P)$ such that $p(\gamma) = c$ and $\theta\gamma = \rho$.

Since θ is cartesian, we obtain that γ coequalises d_0^P and d_1^P, so, since ρ is a coequaliser, there exists a unique morphism $\sigma : \pi_0(P) \to t^*\pi_0(P)$ such that $\sigma\rho = \gamma$.

From the above, $\theta\sigma\rho = \theta\gamma = \rho$, and, since ρ is an epimorphism, we obtain that

$$\theta\sigma = \mathrm{id}\,.$$

Thus, $\theta\sigma\theta = \theta$, and, since θ is cartesian, in order to show that

$$\sigma\theta = \mathrm{id},$$

it suffices to verify that $p(\sigma\theta) = \mathrm{id}$. Indeed, $p(\sigma\theta)c = p(\sigma)tc = p(\sigma)p(\rho) = p(\sigma\rho) = p(\gamma) = c$, and the conclusion follows since c is an epimorphism. □

part-adj-pi0

Proposition 2.11. *With notation and assumptions of 2.10, the pullback functor*

$$\eta_{\mathbb{C}}^* : \mathscr{P}(\pi_0(\mathbb{C})) \simeq \mathscr{P}^{\mathbf{\Delta}(\pi_0(\mathbb{C}))} \to \mathscr{P}^{\mathbb{C}}$$

induced by the unique precategory morphism

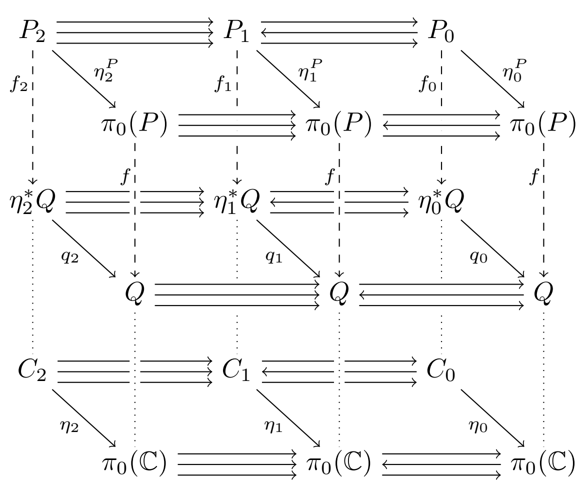
$$\eta_{\mathbb{C}} : \mathbb{C} \to \mathbf{\Delta}(\pi_0(\mathbb{C}))$$

determined by $\eta_0 = c$ has a partial left adjoint π_0 defined at P, i.e., we have a bijection

$$\mathscr{P}(\pi_0(\mathbb{C}))(\pi_0(P), Q) \simeq \mathscr{P}^{\mathbb{C}}(P, \eta_{\mathbb{C}}^* Q),$$

natural in $Q \in \mathscr{P}(\pi_0(\mathbb{C}))$, and $P \in \mathscr{P}^{\mathbb{C}}$, whenever $\pi_0(P)$ exists.

Proof. The assumptions yield the diagram



without the dashed arrows, where all horizontal arrows in $\mathscr{P}$ except possibly η_0^P, η_1^P and η_2^P are cartesian.

By construction, q_0 coequalises the source and the target morphisms of η^*Q, whence, if we take a morphism $(f_0, f_1, f_2) \in \mathscr{P}^{\mathbb{C}}(P, \eta^*Q)$, $q_0 f_0$ coequalises d_0^P, d_1^P, so there exists a unique morphism $f : \pi_0(P) \to Q$ that makes the rightmost vertical square, and ultimately the whole diagram, commutative.

Conversely, if we start with a morphism $f : \pi_0(P) \to Q$, using that q_0 is cartesian, we obtain a unique morphism $f_0 : P_0 \to \eta_0^* Q$ that makes the rightmost vertical square commutative. Since the composite $\eta_1^*Q \to \eta_0^*Q \to Q$ is cartesian, we obtain a unique morphism $f_1 : P_1 \to \eta_1^*Q$ which makes the square involving $P_1, \eta_1^*Q, Q, \pi_0(P)$ commutative, and, given that q_0 is cartesian, f_1 and f_0 commute with source and target morphisms. Using that q_1 is cartesian, f_1 and f_0 commute with the identity sections. Continuing in the same fashion, we obtain a unique $f_2 : P_2 \to \eta_2^*Q$ that makes the whole diagram commutative. □

2.12. **Precategorical descent.**

def:precat-desc

Definition 2.13. Let
$$f : \mathbb{C} \to \mathbb{D}$$
be a morphism in $\mathbf{PreCat}(\mathscr{C})$. We say that

(1) f is a *descent morphism* for $\mathscr{P}$, if the functor $f^* : \mathscr{P}^{\mathbb{D}} \to \mathscr{P}^{\mathbb{C}}$ is fully faithful, and
(2) f is of *effective descent* for $\mathscr{P}$, if the functor $f^* : \mathscr{P}^{\mathbb{D}} \to \mathscr{P}^{\mathbb{C}}$ is an equivalence of categories.

univ-ce-desc

Lemma 2.14. *Suppose that the reflexive coequaliser*
$$C_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} C_0 \longrightarrow \pi_0(\mathbb{C})$$
exists in $\mathscr{C}$.

The morphism of precategories
$$\eta : \mathbb{C} \to \mathbf{\Delta}(\pi_0(\mathbb{C}))$$
is a descent morphism for $\mathscr{P}$ if and only if the above reflexive coequaliser is universal for $\mathscr{P}$, i.e., for every $Q \in \mathscr{P}(\pi_0(\mathbb{C}))$, the diagram
$$\eta_1^*Q \rightrightarrows \eta_0^*Q \longrightarrow Q$$
remains a coequaliser.

Proof. The above diagram is a coequaliser if and only if $\pi_0(\eta^*Q) \simeq Q$, making η^* fully faithful. □

2.15. **Descent data.**

def-dd

Definition 2.16. Given a family $\mathscr{U} = \{U_i \xrightarrow{u_i} U : i \in I\}$ of morphisms in $\mathscr{C}$, the *category of descent data*
$$\mathrm{DD}(\mathscr{U})$$
consists of tuples
$$\left((P_i)_{i\in I}, (\varphi_{ij})_{(i,j)\in I^2}\right),$$
where

- for every $i \in I$, $P_i \in \mathscr{P}(U_i)$;

- for every $i, j \in I$, $\varphi_{ij} : \pi^*_{ij,i} P_i \to \pi^*_{ij,j} P_j$ is an isomorphism in $\mathscr{P}(U_{ij})$, where $U_{ij} = U_i \times_U U_j$ and the corresponding projections are given by the pullback diagram

$$\begin{array}{ccccc} & & U_{ij} & & \\ & \overset{\pi_{ij,i}}{\swarrow} & & \overset{\pi_{ij,j}}{\searrow} & \\ U_i & & & & U_j \\ & \underset{u_i}{\searrow} & & \underset{u_j}{\swarrow} & \\ & & U & & \end{array}$$

satisfying the *cocycle condition*: for every $i, j, k \in I$, considering the triple pullback $U_{ijk} = U_i \times_U U_j \times_U U_k$ and the corresponding projections

$$\begin{array}{ccccc} & & U_{ijk} & & \\ & \swarrow & \downarrow & \searrow & \\ U_{ij} & & U_{ik} & & U_{jk} \\ \downarrow & \searrow\swarrow & & \searrow\swarrow & \downarrow \\ U_i & & U_j & & U_k \\ & \underset{u_i}{\searrow} & \downarrow u_j & \underset{u_k}{\swarrow} & \\ & & U & & \end{array}$$

the diagram

$$\begin{array}{ccc} \pi^*_{ijk,i} P_i & \xrightarrow{\pi^*_{ijk,ik}\varphi_{ik}} & \pi^*_{ijk,k} P_k \\ & {\scriptstyle \pi^*_{ijk,ij}\varphi_{ij}} \searrow \quad \nearrow {\scriptstyle \pi^*_{ijk,jk}\varphi_{jk}} & \\ & \pi^*_{ijk,j} P_j & \end{array}$$

commutes in $\mathscr{P}(U_{ijk})$.

A $\mathrm{DD}(\mathscr{U})$-*morphism*

$$((P_i)_i, (\varphi_{ij})_{ij}) \to \big((P'_i)_i, (\varphi'_{ij})_{ij}\big)$$

consists of a family of morphisms $(P_i \overset{\psi_i}{\to} P'_i)_{i \in I}$ in $\mathscr{P}(U_i)$ such that the diagrams

$$\begin{array}{ccc} \pi^*_{ij,i} P_i & \xrightarrow{\pi^*_{ij,i}\psi_i} & \pi^*_{ij,i} P'_i \\ {\scriptstyle \varphi_{ij}} \downarrow & & \downarrow {\scriptstyle \varphi'_{ij}} \\ \pi^*_{ij,j} P_j & \xrightarrow{\pi^*_{ij,j}\psi_j} & \pi^*_{ij,i} P'_j \end{array}$$

commute in $\mathscr{P}(U_{ij})$.

2.17. **Pullback of descent data.**

Definition 2.18. Let $\mathscr{U} = \{U_i \overset{u_i}{\to} U : i \in I\}$ and $\mathscr{V} = \{V_j \overset{v_j}{\to} V : j \in J\}$ be two families in $\mathscr{C}$.

A *morphism* $\gamma : \mathscr{U} \to \mathscr{V}$ is given by a map of indices $\alpha : I \to J$, a morphism $g : U \to V$ in $\mathscr{C}$ and a family of commutative diagrams

$$\begin{array}{ccc} U_i & \xrightarrow{g_i} & V_{\alpha(i)} \\ {\scriptstyle u_i}\downarrow & & \downarrow{\scriptstyle v_{\alpha(i)}} \\ U & \xrightarrow{g} & V \end{array}$$

for $i \in I$.

If $\gamma' : (\alpha', g', g'_i)$ is another morphism $\mathscr{U} \to \mathscr{V}$ with $g = g'$, we say that morphisms γ and γ' are *homotopy equivalent* and write

$$\gamma \sim \gamma'.$$

pbdesc **Lemma 2.19** (Pullback of descent data, stacks-project [30, 8.3.3])**.** *A morphism $\gamma : \mathscr{U} \to \mathscr{V}$ as above gives a functor*

$$\gamma^* : \mathrm{DD}(\mathscr{V}) \to DD(\mathscr{U}),$$
$$(P_j, \varphi_{jj'}) \mapsto (g_i^* P_{\alpha(i)}, (g_i \times g_{i'})^* \varphi_{\alpha(i)\alpha(i')}).$$

Moreover, if $\gamma \sim \gamma'$, their associated pullback functors are canonically isomorphic,

$$\gamma^* \simeq \gamma'^*.$$

def-effdesc **Definition 2.20.** With notation from 2.19, we say that

(1) $\gamma : \mathscr{U} \to \mathscr{V}$ is a *descent morphism*, if the functor $\gamma^* : \mathrm{DD}(\mathscr{V}) \to \mathrm{DD}(\mathscr{U})$ is fully faithful, and
(2) $\gamma : \mathscr{U} \to \mathscr{V}$ is of *effective descent*, if $\gamma^* : \mathrm{DD}(\mathscr{V}) \to \mathrm{DD}(\mathscr{U})$ establishes an equivalence of categories.

Remark 2.21. Given a morphism $f : U' \to U$ in $\mathscr{C}$, the above definition applied to the morphism of families $f_\square$ given by the diagram

$$\begin{array}{ccc} U & \xleftarrow{f} & U' \\ {\scriptstyle \mathrm{id}}\downarrow & f_\square & \downarrow{\scriptstyle f} \\ U & \xleftarrow{\mathrm{id}} & U \end{array}$$

tells us that $f_\square$ is a morphism of descent (resp., effective descent) if the functor

$$f_\square^* : \mathscr{P}(U) = \mathrm{DD}(\mathrm{id}_U) \to \mathrm{DD}(f)$$

is fully faithful (resp., an equivalence).

Hence, f is a morphism of descent/effective descent in the classical sense whenever $f_\square$ is descent/effective descent in the sense of 2.20.

def-horbox **Definition 2.22** (Horizontal composition of boxes)**.** The horizontal composition of boxes φ and ψ in $\mathscr{C}$

$$\begin{array}{ccccc} U & \xleftarrow{f} & V & \xleftarrow{g} & W \\ {\scriptstyle u}\downarrow & \varphi & \downarrow{\scriptstyle v} & \psi & \downarrow{\scriptstyle w} \\ U' & \xleftarrow{f'} & V' & \xleftarrow{g'} & W' \end{array}$$

is the box $\varphi \circ \psi$ given by the diagram

$$\begin{array}{ccc} U & \xleftarrow{f \circ g} & W \\ {\scriptstyle u}\downarrow & \varphi \circ \phi & \downarrow{\scriptstyle w} \\ U' & \xleftarrow{f' \circ g'} & W' \end{array}$$

rem-horbox *Remark* 2.23. With notation of 2.22, as an immediate consequence of the definition of pullback functors from 2.19, we obtain an isomorphism of functors

$$(\varphi \circ \psi)^* \simeq \psi^* \circ \varphi^*.$$

def-vertbox **Definition 2.24** (Vertical composition of boxes)**.** The vertical composition of boxes φ and φ' in $\mathscr{C}$

$$\begin{array}{ccc} U & \xleftarrow{f} & V \\ {\scriptstyle u}\downarrow & \varphi & \downarrow{\scriptstyle v} \\ U' & \xleftarrow{f'} & V' \\ {\scriptstyle u'}\downarrow & \varphi' & \downarrow{\scriptstyle v'} \\ U'' & \xleftarrow{f''} & V'' \end{array}$$

is the box $\varphi' * \varphi$ given by the diagram

$$\begin{array}{ccc} U & \xleftarrow{f} & V \\ {\scriptstyle u' \circ u}\downarrow & \varphi' * \varphi & \downarrow{\scriptstyle v' \circ v} \\ U'' & \xleftarrow{f''} & V'' \end{array}$$

lem-vertbox **Lemma 2.25.** *With the notation of 2.24, if φ is a descent morphism, and*

$$(f \times f)^* : \mathscr{P}(U \times_{U''} U) \to \mathscr{P}(V \times_{V''} V)$$

is faithful, then $\varphi' * \varphi$ is a descent morphism.

Proof. The boxes

$$\begin{array}{ccc} U & \xleftarrow{\mathrm{id}} & U \\ {\scriptstyle u' \circ u}\downarrow & \rho_U & \downarrow{\scriptstyle u} \\ U'' & \xleftarrow{u'} & U' \end{array} \qquad \begin{array}{ccc} V & \xleftarrow{\mathrm{id}} & V \\ {\scriptstyle v' \circ v}\downarrow & \rho_V & \downarrow{\scriptstyle v} \\ V'' & \xleftarrow{v'} & V' \end{array}$$

satisfy

$$(\varphi' * \varphi) \circ \rho_V = \rho_U \circ \varphi,$$

so we obtain a diagram of categories

$$\begin{array}{ccc} \mathrm{DD}(u' \circ u) & \xrightarrow{(\varphi' * \varphi)^*} & \mathrm{DD}(v' \circ v) \\ {\scriptstyle \rho_U^*}\downarrow & & \downarrow{\scriptstyle \rho_V^*} \\ \mathrm{DD}(u) & \xrightarrow{\varphi^*} & \mathrm{DD}(v) \end{array}$$

where the vertical arrows are faithful.

Indeed, ρ_U^* takes objects $(P,\alpha)\in \mathrm{DD}(u'\circ u)$ to $(P,(\mathrm{id}\times\mathrm{id})^*\alpha)=(P,(U\times_{U'}U\to U\times_{U''}U)^*\alpha)$, and it acts on morphisms $p:(P,\alpha)\to(P',\alpha')$ as identity, hence it is faithful. A similar argument applies to ρ_V^*.

By assumption, the bottom arrow φ^* is also faithful, and it follows that the top arrow $(\varphi'*\varphi)^*$ is too.

It suffices to verify that $(\varphi'*\varphi)^*$ is full. Let $(P,\alpha),(P',\alpha')\in\mathrm{DD}(u'\circ u)$, let $(Q,\beta)=(\varphi'*\varphi)^*(P,\alpha)$, $(Q',\beta')=(\varphi'*\varphi)^*(P',\alpha')$, and let $q:(Q,\beta)\to(Q',\beta')$ be a morphism in $\mathrm{DD}(v'\circ v)$, given by a morphism $q:Q\to Q'$ in $\mathscr{P}(V)$ such that the diagram

$$\begin{array}{ccc} \pi_1^*Q & \xrightarrow{\pi_1^*q} & \pi_1^*Q' \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\beta'} \\ \pi_2^*Q & \xrightarrow{\pi_2^*q} & \pi_2^*Q' \end{array}$$

commutes in $\mathscr{P}(V\times_{V''}V)$.

Since φ^* is fully faithful, there exists a unique morphism $p:\rho_U^*(P,\alpha)\to\rho_U^*(P',\alpha')$ such that $\varphi^*(p)=\rho_V^*(q)$, i.e., a morphism $p:P\to P'$ in $\mathscr{P}(U)$ with $f^*(p)=q$.

We claim that p is a morphism $(P,\alpha)\to(P',\alpha')$ in $\mathrm{DD}(u'\circ u)$, i.e., that the diagram

$$\begin{array}{ccc} \pi_1^*P & \xrightarrow{\pi_1^*p} & \pi_1^*P' \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha'} \\ \pi_2^*P & \xrightarrow{\pi_2^*p} & \pi_2^*P' \end{array}$$

commutes in $\mathscr{P}(V\times_{V''}V)$. This is indeed the case, since pulling the diagram back to $\mathscr{P}(V\times_{V''}V)$ via the faithful functor $(f\times f)^*$ gives the above diagram for q, which is commutative.

□

secteff **Lemma 2.26** (A morphism admitting a section is of effective descent)**.** *Suppose we have a morphism $f:U'\to U$ admitting a section $s:U\to U'$, so that $f\circ s=\mathrm{id}_U$. Then f is of effective descent. More explicitly, the functor*

$$f^*=f_\square^*:\mathscr{P}(U)\to\mathrm{DD}(f)$$

has a quasi-inverse σ^* associated to the box

$$\begin{array}{ccc} U' & \xleftarrow{s} & U \\ {\scriptstyle f}\downarrow & \sigma & \downarrow{\scriptstyle id} \\ U & \xleftarrow{\mathrm{id}} & U \end{array}$$

Proof. The composite $\sigma\circ f_\square$ represents the outer box of the diagram

$$\begin{array}{ccc} U' & \overset{s\circ f}{\underset{\mathrm{id}}{\Longleftarrow}} & U' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ U & \xleftarrow{\mathrm{id}} & U \end{array}$$

and let us denote the inner box by id_f. These boxes are homotopy equivalent in the sense of the 'moreover' clause of 2.19, whence

$$f_\square^*\sigma^* \simeq (\sigma \circ f_\square)^* \simeq \mathrm{id}_f^* = \mathrm{id} : \mathrm{DD}(f) \to \mathrm{DD}(f).$$

Conversely, using the fact that s is a section of f,

$$\sigma^* f_\square^* \simeq (f_\square \circ \sigma)^* = \mathrm{id} : \mathrm{DD}(\mathrm{id}_U) \to \mathrm{DD}(\mathrm{id}_U).$$

□

2.27. **Classical and precategorical descent.** Let $u : U' \to U$ be a morphism in $\mathscr{C}$, and consider the groupoid

$$\mathbb{G}_u$$

associated to the kernel pair of u, as in 2.3

Comparing 2.6 and 2.16, we see that we have an isomorphism

$$\mathrm{DD}_{\mathscr{P}}(u) = \mathscr{P}^{\mathbb{G}_u}.$$

Moreover, a box

$$\begin{array}{ccc} U' & \xrightarrow{f'} & V' \\ {\scriptstyle u}\downarrow & \varphi & \downarrow{\scriptstyle v} \\ U & \xrightarrow{f} & V \end{array}$$

induces a morphism

$$F = (f' \times f' \times f', f' \times f', f') : \mathbb{G}_v \to \mathbb{G}_u,$$

and the functors

$$\varphi^* : \mathrm{DD}(u) \to \mathrm{DD}(v), \quad \text{and} \quad F^* : \mathscr{P}^{\mathbb{G}_u} \to \mathscr{P}^{\mathbb{G}_v}$$

are isomorphic.

Hence, φ is a morphism of descent/effective descent in the classical sense of 2.20 if and only if F is a morphism of descent/effective precategorical descent in the sense of 2.13.

In particular, a morphism $u : U' \to U$ is a morphism of descent/effective descent in the classical sense if and only if $u_\square$ is of descent/effective descent, if and only if the precategory morphism $\mathbb{G}_u \to \mathbf{\Delta}(U) = \mathbb{G}_{\mathrm{id}_U}$ is a morphism of descent/effective precategorical descent.

2.28. **Descent of precategory actions.**

prop:desc-precat

Proposition 2.29. *Let $\mathscr{P} : \mathscr{C}^{\mathrm{op}} \to \mathbf{Cat}$ be a pseudofunctor, and consider the pseudofunctor*

$$\tilde{\mathscr{P}} : \mathbf{PreCat}(\mathscr{C})^{\mathrm{op}} \to \mathbf{Cat}, \quad \mathbb{X} \mapsto \mathscr{P}^{\mathbb{X}}.$$

Let $f : \mathbb{C} \to \mathbb{D} \in \mathbf{PreCat}(\mathscr{C})$ be a morphism of precategories in $\mathscr{C}$ such that

(1) f_0 is of effective descent for $\mathscr{P}$;
(2) f_1 is descent morphism for $\mathscr{P}$;
(3) f_2^* is faithful.

Then f is of effective descent for $\tilde{\mathscr{P}}$.

Proof. We must show that the canonical morphism

$$\mathscr{P}^{\mathbb{D}} = \tilde{\mathscr{P}}(\mathbb{D}) \to \tilde{\mathscr{P}}^{\mathbb{G}_f} = \mathrm{DD}_{\tilde{\mathscr{P}}}(f)$$

is an equivalence of categories, where

$$\mathbb{G}_f \in \mathbf{PreCat}(\mathbf{PreCat}(\mathscr{C}))$$

is given by

$$\mathbb{G}_2 = \mathbb{C} \times_{\mathbb{D}} \mathbb{C} \times_{\mathbb{D}} \mathbb{C} \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; \mathbb{G}_1 = \mathbb{C} \times_{\mathbb{D}} \mathbb{C} \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}\; \mathbb{G}_0 = \mathbb{C}$$

with $\mathbb{G}_0, \mathbb{G}_1, \mathbb{G}_2 \in \mathbf{PreCat}(\mathscr{A})_{/\mathbb{D}}$. Expanding the components of these precategories as columns, we obtain a diagram

$$\begin{array}{ccccc}
C_2 \times_{D_2} C_2 \times_{D_2} C_2 & \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} & C_2 \times_{D_2} C_2 & \substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow} & C_2 \\
\downarrow\downarrow\downarrow & & \downarrow\downarrow\downarrow & & \downarrow\downarrow\downarrow \\
C_1 \times_{D_1} C_1 \times_{D_1} C_1 & \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} & C_1 \times_{D_1} C_1 & \substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow} & C_1 \\
\downarrow\uparrow\downarrow & & \downarrow\uparrow\downarrow & & \downarrow\uparrow\downarrow \\
C_0 \times_{D_0} C_0 \times_{D_0} C_0 & \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} & C_0 \times_{D_0} C_0 & \substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow} & C_0
\end{array}$$

where the rows are the groupoids $\mathbb{G}_{f_2}$, $\mathbb{G}_{f_1}$, $\mathbb{G}_{f_0}$ associated to kernel pairs of morphisms f_2, f_1, f_0 that constitute f.

An action $P \in \tilde{P}^{\mathbb{G}_f}$ consists of a diagram

$$P_2 \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; P_1 \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}\; P_0$$

consisting of $P_2 \in \tilde{\mathscr{P}}(\mathbb{G}_2) = \mathscr{P}^{\mathbb{G}_2}$, $P_1 \in \tilde{\mathscr{P}}(\mathbb{G}_1) = \mathscr{P}^{\mathbb{G}_1}$, $P_0 \in \tilde{\mathscr{P}}(\mathbb{G}_0) = \mathscr{P}^{\mathbb{G}_0}$ and cartesian arrows in the fibration associated to $\tilde{\mathscr{P}}$. Expanding the components of P_2, P_1 and P_0 as columns, we obtain a diagram

$$\begin{array}{ccccc}
P_{2,2} & \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} & P_{1,2} & \substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow} & P_{0,2} \\
\downarrow\downarrow\downarrow & & \downarrow\downarrow\downarrow & & \downarrow\downarrow\downarrow \\
P_{2,1} & \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} & P_{1,1} & \substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow} & P_{0,1} \\
\downarrow\uparrow\downarrow & & \downarrow\uparrow\downarrow & & \downarrow\uparrow\downarrow \\
P_{2,0} & \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} & P_{1,0} & \substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow} & P_{0,0}
\end{array}$$

in the fibred category associated to $\mathscr{P}$, with all morphisms cartesian.

Hence, the rows yield actions $\bar{P}_2 \in \mathscr{P}^{\mathbb{G}_{f_2}} \simeq \mathrm{DD}_{\mathscr{P}}(f_2)$, $\bar{P}_1 \in \mathscr{P}^{\mathbb{G}_{f_1}} \simeq \mathrm{DD}_{\mathscr{P}}(f_1)$, $\bar{P}_0 \in \mathscr{P}^{\mathbb{G}_{f_0}} \simeq \mathrm{DD}_{cP}(f_0)$. Considering descent data as a pseudofunctor on the arrow category

$$\mathrm{DD}_{\mathscr{P}} : \mathrm{Ar}(\mathscr{C})^{\mathrm{op}} \to \mathbf{Cat},$$

the diagram

$$\bar{P}_2 \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; \bar{P}_1 \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}\; \bar{P}_0$$

determines an action

$$\bar{P} \in \mathrm{DD}^{f}_{\mathscr{P}}$$

where $f = (f_2, f_1, f_0) \in \mathbf{PreCat}(\mathrm{Ar}(\mathscr{C}))$.

Hence, we have shown that

$$\mathrm{DD}_{\bar{\mathscr{P}}}(f) \simeq \mathrm{DD}^f_{\mathscr{P}}.$$

Since f_0 is effective descent, there is an object $Q_0 \in \mathscr{P}(D_0)$ such that, writing $\bar{Q}_0 = (Q_0, \mathrm{id})$ for the trivial descent datum, we have $\bar{P}_0 \simeq f^*_{0\square}\bar{Q}_0$. The action isomorphism $d^*_{0,f}\bar{P}_0 \xrightarrow{\alpha} d^*_{1,f}\bar{P}_0$ yields an isomorphism

$$f^*_{1\square}\bar{d}^*_0\bar{Q}_0 \simeq d^*_{0,f}f^*_{0\square}\bar{Q}_0 \simeq d^*_{0,f}\bar{P}_0 \xrightarrow{\alpha} d^*_{1,f}\bar{P}_0 \simeq d^*_{1,f}f^*_{0\square}\bar{Q}_0 \simeq f^*_{1\square}\bar{d}^*_1\bar{Q}_0,$$

where we wrote $\bar{d}_0$ and $\bar{d}_1$ for the obvious boxes/morphisms $\mathrm{id}_{D_1} \to \mathrm{id}_{D_0}$ in $\mathrm{Ar}(\mathscr{C})$. Given that f_1 is descent, we obtain a unique action morphism

$$\bar{d}^*_0\bar{Q}_0 \xrightarrow{\bar{\beta}} \bar{d}^*_1\bar{Q}_0$$

such that $f^*_{1\square}\bar{\beta} \simeq \alpha$. Note that $\bar{\beta}$ is uniquely determined by an isomorphism

$$d^*_0 Q_0 \xrightarrow{\beta} d^*_1 Q_0,$$

and it remains to verify that β satisfies the cocycle condition

$$r^*_1\beta \circ r^*_0\beta \simeq m^*\beta$$

in $\mathscr{P}(D_2)$, or, equivalently, that $\bar{r}^*_1\bar{\beta}\circ\bar{r}^*_0\bar{\beta}$ and $\bar{m}^*\bar{\beta}$ agree up to coherence. Applying the functor $f^*_{2\square}$ to both yields

$$f^*_{2\square}\bar{r}^*_1\bar{\beta} \circ f^*_{2\square}\bar{r}^*_0\bar{\beta} \simeq r^*_{1,f}f^*_{1\square}\bar{\beta} \circ \ r^*_{0,f}f^*_{1\square}\bar{\beta} \simeq r^*_{1,f}\alpha \circ \circ \ r^*_{0,f}\alpha$$

and

$$f^*_{2\square}\bar{m}^*\bar{\beta} \simeq m^*_f f^*_{1\square}\bar{\beta} \simeq m^*_f\alpha,$$

which agree up to coherence by the cocycle condition for α. By faithfulness of $f^*_{2\square}$, we obtain the cocycle condition for β, and we have constructed a unique action $(Q_0, \beta) \in \mathscr{P}^{\mathbb{D}}$ (up to isomorphism) that lifts to P, as desired. □

2.30. **Descent for quasi-projective morphisms.**

efdesc-qp

Proposition 2.31. *A scheme morphism $f : X \to Y$ whose codomain Y is the spectrum of a field k is of effective descent for quasi-projective morphisms.*

Proof. Let k' be a finite extension of k with $X(k') \neq \emptyset$. Writing $Y' = \mathrm{Spec}(k')$, we have a pullback diagram

$$\begin{array}{ccc} X & \xleftarrow{g'} & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Y & \xleftarrow{g} & Y' \end{array}$$

where f' admits a section s afforded by a k'-point of X'. The morphism g is finite locally free surjective, hence of effective descent for quasi-projective morphisms by sga1 [15, VIII, 7.7], and so is g' as a base-change of g. Consider the diagram

$$\begin{array}{ccccccc} X & \xleftarrow{g'} & X' & \xleftarrow{s} & Y' & \xleftarrow{f'} & X' \\ {\scriptstyle f}\downarrow & \alpha & \downarrow{\scriptstyle fg'=gf'} & \beta & \downarrow{\scriptstyle g} & \gamma & \downarrow{\scriptstyle gf'} \\ Y & \xleftarrow{\mathrm{id}} & Y & \xleftarrow{\mathrm{id}} & Y & \xleftarrow{\mathrm{id}} & Y \end{array}$$

defining boxes α, β and γ in $\mathscr{C}$.

Writing σ for the box

$$\begin{array}{ccc} X & \xleftarrow{s} & Y' \\ {\scriptstyle f'}\downarrow & \sigma & \downarrow{\scriptstyle id} \\ Y' & \xleftarrow{\mathrm{id}} & Y' \end{array}$$

we directly verify that $\beta = \mathrm{id}_g * \sigma$ and $\gamma = \mathrm{id}_g * f'_{\square}$.

As in the proof of 2.26, we have that $\sigma \circ f'_{\square} \sim \mathrm{id}_{f'}$, so

$$\beta \circ \gamma = (\mathrm{id}_g * \sigma) \circ (\mathrm{id}_g * f'_{\square}) = \mathrm{id}_g * (\sigma \circ f'_{\square}) \sim \mathrm{id}_g * \mathrm{id}_{f'} = \mathrm{id}_{g \circ f'},$$

whence

$$\gamma^* \beta^* \simeq \mathrm{id}\,.$$

Conversely, $\gamma \circ \beta$ yields identity on the nose, so we conclude that γ^* is an equivalence of categories.

Using the fact that g' is of effective descent and that $g' \times g'$ is finite faithfully flat, Lemma 2.25 gives that the functor α^* associated to $\alpha = \mathrm{id}_f * g'_{\square}$ is fully faithful.

We directly verify that $g_{\square} \circ \gamma = f_{\square} \circ \alpha$, whence we obtain a diagram of categories

$$\begin{array}{ccccc} & & \mathrm{DD}(f) & & \\ & {\scriptstyle f^*_{\square}}\nearrow & & \searrow{\scriptstyle \alpha^*} & \\ \mathscr{P}(Y) & & & & \mathrm{DD}(fg') \\ & {\scriptstyle g^*_{\square}}\searrow & & \nearrow{\scriptstyle \gamma^*} & \\ & & \mathrm{DD}(g) & & \end{array}$$

where we know that $\alpha^* f^*_{\square} = \gamma^* g^*_{\square}$ is an equivalence of categories, and α^* is fully faithful, so we deduce that $f^*_{\square}$ is also an equivalence. $\square$

3. Categorical Galois theory

ss:janelidze-gal

3.1. Classical Janelidze's categorical Galois theory. Consider an adjoint pair of functors

$$\begin{array}{c} \mathscr{A} \\ {\scriptstyle S}\big\downarrow \dashv \big\uparrow{\scriptstyle C} \\ \mathscr{X} \end{array}$$

with unit $\eta : \mathrm{id} \to CS$ and counit $\epsilon : SC \to \mathrm{id}$. If $\mathscr{A}$ admits pullbacks, for any $X \in \mathscr{A}$ we obtain an adjunction

$$\begin{array}{c} \mathscr{A}_{/X} \\ {\scriptstyle S_X}\big\downarrow \dashv \big\uparrow{\scriptstyle C_X} \\ \mathscr{X}_{/S(X)} \end{array}$$

where

$$S_X(A \xrightarrow{a} X) = S(A) \xrightarrow{S(a)} S(X),$$

and $C_X(E \xrightarrow{e} S(X))$ is obtained by forming the pullback

$$\begin{array}{ccc} C_X(e) & \longrightarrow & C(E) \\ \big\downarrow & & \big\downarrow{\scriptstyle C(e)} \\ X & \xrightarrow{\eta_X} & CS(X) \end{array}$$

in $\mathscr{A}$.

A morphism $X \xrightarrow{f} Y$ in $\mathscr{A}$ gives rise to the pullback/base change functor

$$f^* : \mathscr{A}_{/Y} \to \mathscr{A}_{/X},$$

which admits a left adjoint

$$f_! : \mathscr{A}_{/X} \to \mathscr{A}_{/Y}, \quad a \mapsto f \circ a.$$

Following borceux-janelidze [8, Def. 5.1.7] an object $A \xrightarrow{a} Y \in \mathscr{A}_{/Y}$ is *split* by $X \xrightarrow{f} Y \in \mathscr{A}$ when the unit $\eta^X : \mathrm{id} \to C_X S_X$ of adjunction $S_X \dashv C_X$ gives an isomorphism

$$\eta^X_{f^*a} : f^*a \to C_X S_X(f^*a).$$

If C_X is fully faithful, a is split by f (borceux-janelidze [8, Cor. 5.1.13]), if and only if there exists an object $E \xrightarrow{e} S(X)$ such that

$$f^*a \simeq C_X(e).$$

We write

$$\mathrm{Split}_Y(f)$$

for the full subcategory of $\mathscr{A}_{/Y}$ of objects split by f.

The morphism f is of *relative Galois descent* if

(1) f^* is monadic;
(2) the counit $\epsilon^X : S_X C_X \to \mathrm{id}$ of adjunction $S_X \dashv C_X$ is an isomorphism;
(3) for every $E \xrightarrow{e} S(X)$ in $\mathscr{X}_{/S(X)}$, the object $f_!\, C_X(e) \in \mathscr{A}_{/Y}$ is split by f.

If $X \xrightarrow{f} Y$ is of relative Galois descent, the Galois precategory

$$\mathrm{Gal}[f] = S(\mathbb{G}_f)$$

is actually an internal groupoid in $\mathscr{X}$ given by the data

$$S(X \times_Y X) \times_{S(X)} S(X \times_Y X) \xrightarrow{(S(\pi_1), S(\pi_4))} S(X \times_Y X) \underset{S(\pi_2)}{\overset{S(\pi_1)}{\underset{\longleftarrow\, S(\Delta)}{\rightrightarrows}}} S(X), \qquad S(\tau) \circlearrowright S(X \times_Y X)$$

where τ is the morphism interchanging the copies of X, and Δ is the diagonal.

Janelidze's *Galois theorem* (borceux-janelidze [8, Thm. 5.1.24]) gives an equivalence of categories

$$\mathrm{Split}_Y(f) \simeq \mathscr{X}^{\mathrm{Gal}[f]}.$$

of f-split objects and the actions of the internal groupoid $\mathrm{Gal}[f]$ in $\mathscr{X}$, as in 2.5.

The proof consists in verifying that the monad $\mathbb{T}$ associated to the adjunction

$$\begin{array}{c} \mathrm{Split}_Y(f) \\ F = f_!\, C_X \uparrow \dashv \downarrow U = S_X\, f^* \\ \mathscr{X}_{/S(X)} \end{array}$$

of the monadic functor U ([8, Cor. 5.1.21]), with functorial part $T = UF$, is isomorphic to the monad $\mathbb{T}'$ on $\mathscr{X}_{/S(X)}$ associated to the adjunction

$$\begin{array}{c}\mathscr{X}^{\mathrm{Gal}[f]}\\ F' \left(\dashv \right) U' \\ \mathscr{X}_{/S(X)}\end{array}$$

where U' is the forgetful functor omitting the groupoid action, and F' is the 'representable internal diagram' functor, whose functorial part is $T' = d_{1!}d_0^*$ and the category of algebras is the category $\mathscr{X}^{\mathrm{Gal}[f]}$. Hence, we obtain equivalences

$$\mathrm{Split}_Y[f] \xrightarrow{K^{\mathbb{T}}} (\mathscr{X}_{/S(X)})^{\mathbb{T}} \simeq (\mathscr{X}_{/S(X)})^{\mathbb{T}'} \xleftarrow{K^{\mathbb{T}'}} \mathscr{X}^{\mathrm{Gal}[f]},$$

where we wrote $K^{\mathbb{T}}$ and $K^{\mathbb{T}'}$ for the comparison functors of the respective monads.

In this case, modulo the identification of the category of $\mathbb{T}$-algebras $(\mathscr{X}_{/S(X)})^{\mathbb{T}}$ with $\mathscr{X}^{\mathrm{Gal}[f]}$, the functors realising the sought-after equivalence are the comparison functor

$$\Phi = K^{\mathbb{T}} : \mathrm{Split}_Y(f) \xrightarrow{\sim} (\mathscr{X}_{/S(X)})^{\mathbb{T}}, \quad p \mapsto (U(p), U(\varepsilon_p)),$$

and its left adjoint

$$\Psi : (q, \nu) \mapsto \mathrm{Coeq}\,(FUF(q) \underset{\varepsilon_{Fq}}{\overset{F(\nu)}{\rightrightarrows}} F(q)),$$

where $(Q \xrightarrow{q} X_0, \nu)$ is a $\mathbb{T}$-algebra and $\varepsilon : FU \to \mathrm{id}$ is the counit of the adjunction $F \dashv U$, and the coequaliser exists by the proof of Beck's monadicity criterion as in [4, 3.14].

By identifying $\mathbb{T}$ and $\mathbb{T}'$, and writing $\mathrm{Gal}[f] = (S(X \times_Y X), S(X)) = (G_1, X_0)$, the top arrow appearing in the coequaliser is obtained by applying F to the action $G_1 \times_{X_0} Q \xrightarrow{\nu} Q$, which, modulo the identification $X \times_{X_0} (G_1 \times_{X_0} Q) \simeq X \times_{X_0} G_1 \times_{X_0} Q$ gives the morphism

$$\mathrm{id} \times \nu : X \times_{X_0} G_1 \times_{X_0} Q \to X \times_{X_0} Q.$$

If $p \in \mathrm{Split}_Y[f]$ then $f^*(p) \simeq C_X(q)$ for some q, whence the counit ε_p is

$$FUp = f_! C_X S_X f^* p \simeq f_! C_X S_X C_X q \simeq f_! C_X q \simeq f_! f^* p \to p,$$

so the bottom arrow ε_{Fq} identifies with $f_! f^* Fq \to Fq$, which, modulo the isomorphism

$$X \times_Y (X \times_{X_0} Q) \simeq (X \times_Y X) \times_X (X \times_{X_0} Q) \simeq (X \times_{X_0} G_1) \times_X (X \times_{X_0} Q) \simeq X \times_{X_0} G_1 \times_{X_0} Q,$$

identifies with

$$\mu \times \mathrm{id} : X \times_{X_0} G_1 \times_{X_0} Q \to X \times_{X_0} Q,$$

where μ denotes the $\mathrm{Gal}[f]$-action on X. Thus, we may symbolically write the above coequaliser as the quotient

$$\Psi(q, \nu) = \mathrm{Coeq}(\mathrm{id} \times \nu, \mu \times \mathrm{id}) = F(q)/\,\mathrm{Gal}[f] = (X \times_{X_0} Q)/\,\mathrm{Gal}[f],$$

by the twisted-diagonal action of the Galois groupoid on $X \times_{X_0} Q$.

3.2. **Carboni-Magid-Janelidze Galois correspondence.**

Fact 3.3 (carboni [10]). *Let $\mathscr{C}$ be a category with pullbacks and coequalisers of equivalence relations. In the presence of pullbacks, regular epimorphisms coincide with effective epimorphisms, and we call them* quotients *for short.*

Let $G = (G_1, G_0, d_0, d_1, e, m)$ be an internal groupoid in $\mathscr{C}$. Let us write $\tilde{G} \in \mathscr{C}^G$ for the canonical action of G on itself. There is a bijection

$$\mathrm{Sub}(G) \simeq \mathrm{Equiv}(\tilde{G})$$

between the set of subgroupoids of G with the same object of objects and the set of equivalence relations on $\tilde{G}$ in $\mathscr{C}^G$ as follows.

Given a subgroupoid $G' = (G'_1, G_0)$ with $\iota : G'_1 \hookrightarrow G_1$, the corresponding equivalence relation on $\tilde{G}$ is

$$R_{G'} = G_1 \times_{G_0} G'_1 \underset{m(\mathrm{id}_{G_1} \times \iota)}{\overset{\pi_1}{\rightrightarrows}} G_1$$

Conversely, if $R \hookrightarrow \tilde{G} \times \tilde{G}$ is an equivalence relation on $\tilde{G}$ in $\mathscr{C}^G$, we define the corresponding subgroupoid by the pullback

$$\begin{array}{ccc} G'_1 & \longrightarrow & R \\ \big\downarrow & & \big\downarrow \\ G_1 & \xrightarrow{(\mathrm{id}_{G_1}, ed_1)} & G_1 \times_{G_0} G_1 \end{array}$$

where $G_1 \times_{G_0} G_1$ is the kernel pair of d_1.

A subgroupoid is called effective *is the associated equivalence relation is effective in the sense that it is the kernel pair of its coequaliser.*

th-cmj-corr **Theorem 3.4.** *With notation from 3.1, suppose that $f : X \to Y$ is of relative Galois descent with Galois groupoid $G = \mathrm{Gal}[f]$ internal in $\mathscr{X}$. There is an anti-isomorphism*

$$\mathrm{SplitQuo}[f] \simeq \mathrm{EffSub}(\mathrm{Gal}[f])$$

that assigns

$$\begin{array}{ccc} \left(\begin{array}{ccc} & X & \\ \swarrow & & \big\downarrow f \\ P & \xrightarrow{p} & Y \end{array}\right) & \mapsto & \mathrm{Gal}[X \to P] \\ & & \\ X/G' & \leftarrow\!\shortmid & G' \end{array}$$

between the ordered set of quotients of X over Y in Split$[f]$ *and the ordered set of effective subgroupoids of* Gal$[f]$.

Proof. We use the equivalence established by functors Φ and Ψ from 3.1 and the fact that, in the presence of pullbacks, quotients (regular epimorphisms) agree with effective epimorphisms.

If the quotient p of X from the above diagram is f-split by $Q \xrightarrow{q} X_0 = S(X)$, i.e., $f^*p \simeq C_X(q)$, applying the comparison functor Φ gives an effective quotient

$$\begin{array}{ccc} & & \tilde{G} \\ & \swarrow & \downarrow \\ Q & & \\ & \underset{q}{\searrow} & \downarrow \\ & & X_0 \end{array}$$

and an effective equivalence relation $\tilde{G} \times_Q \tilde{G} \hookrightarrow \tilde{G} \times_{X_0} \tilde{G}$ on $\tilde{G}$. The corresponding effective subgroupoid G_P of G is given by

$$\begin{array}{ccc} G_{P,1} & \longrightarrow & G_1 \times_Q G_1 \\ \big\downarrow & & \big\downarrow \\ G_1 & \xrightarrow{(\mathrm{id}_{G_1}, ed_1)} & G_1 \times_{G_0} G_1 \end{array}$$

Conversely, an effective subgroupoid G' of G is associated with the action

$$\tilde{G}/G' = \mathrm{Coeq}(R_{G'} \rightrightarrows \tilde{G})$$

in $\mathscr{X}^G$ given as the coequaliser of its associated effective equivalence relation, and then taken to the split quotient

$$\begin{aligned} \Psi(\tilde{G}/G') &\simeq (X \times \tilde{G}/G')/G \simeq X/G' = f/G' \\ &= \mathrm{Coeq}\left(F(G_1' \to X_0) \to F(G_1 \to X_0) \underset{\mathrm{proj}}{\overset{\mu}{\rightrightarrows}} f\right), \end{aligned}$$

where $\mu : X \times_{X_0} G_1 \to X$ denotes the action of G on X.

The assignments given above clearly establish an equivalence because they are constructed as restrictions of Φ and Ψ to the appropriate full subcategories, but we find it useful to provide a direct proof of the correspondence.

By construction, the groupoid associated to X/G' is

$$G_{X/G'} = G \times_{G \times_{X_0} G} (G \times_{G/G'} G) \simeq G_{G \times_{X_0} G}(G \times_{X_0} G') \simeq G',$$

where the isomorphism holds by effectivity.

Conversely, if P is f-split by Q, then

$$P \simeq \Psi(Q) = \Psi(G/G_P) = (X \times_{X_0} G/G_P)/G \simeq X/G_P.$$

It remains to show that $G_{P,1} \simeq S(X \times_P X)$, i.e.,

$$G_P \simeq \mathrm{Gal}[X \to P].$$

By applying C_X to the pullback diagram defining $G_{P,1}$ above, and using the fact that the right adjoint C_X commutes with pullbacks, as well as the relations witnessing splitting of the objects involved, we obtain a pullback diagram

$$\begin{array}{ccc} C_X(G_{P,1}) & \longrightarrow & f^*(X) \times_{f^*(P)} f^*(X) \\ \big\downarrow & & \big\downarrow \\ f^*(X) & \xrightarrow{(\mathrm{id}, \Delta \circ \mathrm{proj})} & f^*(X) \times_X f^*(X) \end{array}$$

which, using $f^*X \simeq X \times_Y X$ and the groupoid structure of $\mathbb{G}_f$, simplifies to $C_X(G_{P,1}) \simeq X \times_P X$, whence $G_{P,1} \simeq S_X(X \times_P X)$. □

3.5. **Indexed categorical Galois theory.**

indexed-gal-th **Theorem 3.6** ([8, 7.5.3, discussion after 7.6.2]). *Suppose we are given* borceux-janelidze

1. *a functor* $S : \mathscr{A} \to \mathscr{X}$*;*
2. *pseudo-functors* $K : \mathscr{X}^{\mathrm{op}} \to \mathbf{Cat}$ *and* $L : \mathscr{A}^{\mathrm{op}} \to \mathbf{Cat}$*;*
3. *a pseudo-natural transformation* $\alpha : K \circ S \Rightarrow L$*;*
4. *a 'precategorical decomposition' of a morphism* $X \xrightarrow{f} Y$ *in* $\mathscr{A}$*, i.e., a commutative diagram of morphisms of precategories*

$$\begin{array}{ccc} \mathbf{\Delta}(X) & \xrightarrow{\mathbf{\Delta}(f)} & \mathbf{\Delta}(Y) \\ & {\scriptstyle i}\searrow \quad \nearrow{\scriptstyle \pi} & \\ & \mathbb{C} & \end{array}$$

with $i_0 = \mathrm{id}_X$*, and such that the components* α_{C_0}*,* α_{C_1} *and* α_{C_2} *are full and faithful.*

If $(f, (i, \mathbb{C}, \pi))$ *is of effective descent with respect to* L*, we have an equivalence of categories*

$$\mathrm{Split}_\alpha(f) \simeq K^{S \circ \mathbb{C}}.$$

4. Differential algebraic geometry

s:dif-alg

4.1. **Differential schemes.**

A *differential scheme*

$$(X, (\mathscr{O}_X, \delta_X))$$

is a differentially ringed space where $(X, \mathscr{O}_X)$ is a scheme, and $\delta_X \in \mathrm{Der}(\mathscr{O}_X, \mathscr{O}_X)$.

A *morphism of differential schemes*

$$(f, \varphi) : (X, (\mathscr{O}_X, \delta_X)) \to (Y, (\mathscr{O}_Y, \delta_Y))$$

is a morphism of differentially ringed spaces which is also a scheme morphism, i.e., it is a scheme morphism $(f, \varphi) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ whose structure homomorphism is a morphism of differential rings $\varphi : (\mathscr{O}_Y, \delta_Y) \to f_*(\mathscr{O}_X, \delta_X)$, or, equivalently, its mate is a morphism of differential rings $\varphi^\sharp : f^*(\mathscr{O}_Y, \delta_Y) \to (\mathscr{O}_X, \delta_X)$.

They constitute the *category of differential schemes* denoted

$$\delta\text{-Sch}.$$

We have an obvious functor

$$C : \mathrm{Sch} \to \delta\text{-Sch}, \quad (X, \mathscr{O}_X) \mapsto (X, (\mathscr{O}_X, 0))$$

that turns a scheme into a differential scheme with the trivial derivation 0.

Given a scheme morphism $(f, \varphi) : (X, \mathscr{O}_X) \to (S, \mathscr{O}_S)$ and an $\mathscr{O}_X$-module $\mathscr{F}$, we say that an additive morphism $D : \mathscr{O}_X \to \mathscr{F}$ is an S-*derivation* of $\mathscr{O}_X$ to $\mathscr{F}$ if it is an $f^*\mathscr{O}_S$-derivation via $\varphi^\sharp : f^*\mathscr{O}_S \to \mathscr{O}_X$, or, equivalently, if $D_x : \mathscr{O}_{X,x} \to \mathscr{F}_x$ is an $\mathscr{O}_{S,f(x)}$-derivation via $\varphi^\sharp_x : \mathscr{O}_{S,f(x)} \to \mathscr{O}_{X,x}$ for every $x \in X$. The collection of all S-derivations of $\mathscr{O}_X$ to $\mathscr{F}$ is denoted

$$\mathrm{Der}_S(\mathscr{O}_X, \mathscr{F}).$$

A differential scheme $(X, (\mathscr{O}_X, \delta_X))$ equipped with a scheme morphism $(f, \varphi) : (X, \mathscr{O}_X) \to (S, \mathscr{O}_S)$ is called an S-*differential scheme* provided $\delta_X \in \mathrm{Der}_S(\mathscr{O}_X, \mathscr{O}_X)$.

Clearly, an S-differential scheme (X, δ_X) is a morphism of differential schemes $(X, \delta_X) \to C(S) = (S, 0)$. Thus, the category of S-differential schemes is the slice category

$$\delta\text{-Sch}_S = \delta\text{-Sch}_{/C(S)}.$$

4.2. **Differential schemes and vector fields.** Let $(f, \varphi) : X \to S$ be a scheme morphism. By [14, 16.5.3], the universal differential

$$d_{X/S} : \mathscr{O}_X \to \Omega^1_{X/S}$$

is an S-derivation, and composing with $d_{X/S}$ induces an isomorphism of $\Gamma(X, \mathscr{O}_X)$-modules

$$\mathrm{Hom}_{\mathscr{O}_X}(\Omega^1_{X/S}, \mathscr{F}) \xrightarrow{\sim} \mathrm{Der}_S(\mathscr{O}_X, \mathscr{F})$$

for any $\mathscr{O}_X$-module $\mathscr{F}$.

Let

$$T_{X/S} = \mathbf{V}(\Omega^1_{X/S})$$

be the tangent bundle of X relative to S, defined as the vector bundle associated to the quasi-coherent $\mathscr{O}_X$-module $\Omega^1_{X/S}$ (i.e., the spectrum of the quasi-coherent $\mathscr{O}_X$-algebra $\mathrm{Sym}(\Omega^1_{X/S})$) see [14, 16.5.12].

For a point $x \in X$, the *tangent space of X at x relative to S* is defined [14, 16.5.13] as

$$T_{X/S}(x) = \left(T_{X/S} \times_X \mathrm{Spec}(\kappa(x))\right)(\kappa(x)) \simeq \mathrm{Hom}_{\kappa(x)}(\Omega^1_{X/S} \otimes_{\mathscr{O}_X} \kappa(x), \kappa(x)).$$

Choosing an S-derivation $\delta_X \in \mathrm{Der}_S(\mathscr{O}_X, \mathscr{O}_X)$, we therefore obtain a morphism of $\mathscr{O}_X$-modules

$$\Omega^1_{X/S} \to \mathscr{O}_X,$$

which yields a section $X \to T_{X/S}$ of the relative tangent bundle, that we think of as a *vector field* on X relative to S.

Pointwise, for every $x \in X$, pulling back the $\mathscr{O}_X$-modules to $\kappa(x)$-modules via the morphism $\mathrm{Spec}(\kappa(x)) \to X$ yields a morphism

$$\Omega^1_{X/S} \otimes_{\mathscr{O}_X} \kappa(x) \to \kappa(x),$$

an element of $T_{X/S}(x)$. This construction is in line with the classical notion of vector field in differential geometry as a map that sends a point to a vector in the corresponding tangent space.

A point $x \in X$ is a *leaf* for the vector field given by δ_X if the corresponding tangent vector at x is 0.

4.3. **Differential spectra and affine differential schemes.** The spectrum of the underlying ring of a differential ring (A, δ_A) carries a natural structure of a differential scheme

$$\mathrm{Spec}(A, \delta_A) = (\mathrm{Spec}(A), (\mathscr{O}_{\mathrm{Spec}(A)}, \delta_{\mathrm{Spec}(A)})).$$

Indeed,

$$\delta_{\mathrm{Spec}(A)} : \mathscr{O}_{\mathrm{Spec}(A)} \to \mathscr{O}_{\mathrm{Spec}(A)}$$

is determined on basic opens $D(f)$ in $\mathrm{Spec}(A)$, for $f \in A$ by setting

$$\delta_{\mathrm{Spec}(A), D(f)} : \mathscr{O}_{\mathrm{Spec}(A)}(D(f)) = A_f \to A_f = \mathscr{O}_{\mathrm{Spec}(A)}(D(f)), \quad \frac{a}{f} \mapsto \frac{\delta_A(a) f - a \delta_A(f)}{f^2}.$$

This construction extends to a functor

$$\mathrm{Spec} : \delta\text{-}\mathrm{Rng}^{\mathrm{op}} \to \delta\text{-}\mathrm{Sch},$$

right adjoint to the global sections functor

$$\Gamma : (X, (\mathscr{O}_X, \delta_X)) \mapsto (\mathscr{O}_X(X), \delta_{X,X})$$

as the diagram

$$\begin{array}{c} \delta\text{-}\mathrm{Sch} \\ \Gamma \Big\downarrow \dashv \Big\uparrow \mathrm{Spec} \\ \delta\text{-}\mathrm{Rng}^{\mathrm{op}} \end{array}$$

depicts.

A differential scheme is *affine*, if it is isomorphic to a spectrum of a differential ring. Clearly, the category of affine differential schemes is anti-equivalent to the category of differential rings, i.e.,

$$\delta\text{-}\mathrm{Aff} \simeq \delta\text{-}\mathrm{Rng}^{\mathrm{op}}.$$

4.4. **Differential schemes as precategory actions.** Given a scheme S, let us consider the diagram of quasi-coherent $\mathscr{O}_S$-algebras

$$\mathscr{O}_S[\epsilon_0, \epsilon_1]/(\epsilon_0^2, \epsilon_0\epsilon_1, \epsilon_1^2) \underset{\epsilon_1 \leftarrow\!\shortmid\, \epsilon}{\overset{\epsilon_0 \leftarrow\!\shortmid\, \epsilon}{\xleftarrow{\epsilon_0+\epsilon_1 \leftarrow\!\shortmid\, \epsilon}}} \mathscr{O}_S[\epsilon]/(\epsilon^2) \underset{\mathrm{id}+0}{\overset{\mathrm{id}+0}{\xrightarrow{\epsilon \mapsto 0}}} \mathscr{O}_S$$

Applying the spectrum of quasi-coherent $\mathscr{O}_S$-algebras functor EGAII [13, 1.3], we obtain a precategory $\mathbb{D}(S)$

$$S_2 \underset{r_1}{\overset{r_0}{\xrightarrow{m}}} S_1 \underset{d_1}{\overset{d_0}{\xleftarrow{n}}} S_0$$

in $\mathrm{Sch}_{/S}$ consisting of schemes affine over $S_0 = S$, and the underlying morphisms of topological spaces are all identities.

Note, if we write

$$\mathbb{D}(\mathbb{Z}) = \mathbb{D}(\mathrm{Spec}(\mathbb{Z})),$$

then

$$\mathbb{D}(S) = \mathbb{D}(\mathbb{Z}) \times \mathbf{\Delta}(S) = (S \times \mathrm{Spec}(\mathbb{Z}[\epsilon_0, \epsilon_1]/(\epsilon_0^2, \epsilon_0\epsilon_1, \epsilon_1^2)), S \times \mathrm{Spec}(\mathbb{Z}[\epsilon]/(\epsilon^2)), S).$$

diff-sch-precats **Proposition 4.5.** *The category of S-differential schemes is equivalent to the category of $\mathbb{D}(S)$-actions in $\mathrm{Sch}_{/S}$ (cf. 2.5),*

$$\delta\text{-}\mathrm{Sch}_S \simeq (\mathrm{Sch}_{/S})^{\mathbb{D}(S)}.$$

Proof. Using 2.6, an action is determined by a scheme morphism $X_0 = X \to S = S_0$ and an S_1-automorphism $\alpha : X_1 \to X_1$, where $X_1 = X \times_{S_0} S_1$ satisfying $n^*\alpha = \mathrm{id}$ and the cocycle condition. Equivalently, it is given by an $\mathscr{O}_S[\epsilon]/(\epsilon^2)$-automorphism of $\mathscr{O}_{X_1} = \mathscr{O}_X[\epsilon]/(\epsilon^2)$ which, tensored by the augmentation morphism $\eta_S : \mathscr{O}_S[\epsilon]/(\epsilon^2) \to \mathscr{O}_S$ gives $\mathrm{id}_{\mathscr{O}_X}$, and it follows that it must be of the form

$$\mathrm{id}_{\mathscr{O}_X[\epsilon]/(\epsilon^2)} + \epsilon\delta_\alpha \circ \eta_X,$$

where $\delta_\alpha \in \mathrm{Der}_S(\mathscr{O}_X, \mathscr{O}_X)$ and $\eta_X : \mathscr{O}_X[\epsilon]/(\epsilon^2) \to \mathscr{O}_X$ is the augmentation homomorphism. Note that such a morphism always has an inverse $id_{\mathscr{O}_X[\epsilon]/(\epsilon^2)} - \epsilon\delta_\alpha \circ \eta_X$.

The cocycle condition is trivially satisfied/superfluous, since $r_i^*\alpha$ is determined by the automorphism of $\mathscr{O}_X[\epsilon_0,\epsilon_1]/(\epsilon_0^2,\epsilon_0\epsilon_1,\epsilon_1^2)$ given by $\operatorname{id}+\epsilon_i\delta_\alpha\circ\eta_{X,2}$ for $i=0,1$, and $m^*\alpha$ is given by $\operatorname{id}+(\epsilon_0+\epsilon_1)\delta_\alpha\circ\eta_{X,2}$, where we wrote $\eta_{X,2}$ for the augmentation homomorphism $\mathscr{O}_X[\epsilon_0,\epsilon_1]/(\epsilon_0^2,\epsilon_0\epsilon_1,\epsilon_1^2)\to\mathscr{O}_X$, and these morphisms compose in the way prescribed by the cocycle condition.

Thus, an action uniquely determines a derivation, and we obtain an S-differential scheme $(X,(\mathscr{O}_X,\delta_\alpha))$.

Conversely, given an S-differential scheme $(X,(\mathscr{O}_X,\delta_X))$, we can create a $\mathbb{D}(S)$-action from the diagram

$$\mathscr{O}_X[\epsilon_0,\epsilon_1]/(\epsilon_0^2,\epsilon_0\epsilon_1,\epsilon_1^2)\ \begin{array}{c}\xleftarrow{\epsilon_0\,\leftarrow\!\shortmid\,\epsilon}\\ \xleftarrow{\epsilon_0+\epsilon_1\,\leftarrow\!\shortmid\,\epsilon}\\ \xleftarrow{\epsilon_1\,\leftarrow\!\shortmid\,\epsilon}\end{array}\ \mathscr{O}_X[\epsilon]/(\epsilon^2)\ \begin{array}{c}\xleftarrow{\operatorname{id}+0}\\ \xrightarrow{\epsilon\mapsto 0}\\ \xleftarrow{\operatorname{id}+\epsilon\delta_X}\end{array}\ \mathscr{O}_X$$

by applying the spectrum of quasi-coherent $\mathscr{O}_P$-algebras functor. All the resulting morphisms are cartesian, so we obtain an action by 2.7. □

C-for-precat *Remark* 4.6. The scheme of connected components is

$$\pi_0(\mathbb{D}(S))=S$$

because it is calculated as the coequaliser of morphisms d_0,d_1 in $\mathbb{D}(S)$, which agree. Hence, writing

$$\eta:\mathbb{D}(S)\to\boldsymbol{\Delta}(S)$$

for the associated morphism of precategories, the functor

$$C:\mathrm{Sch}_{/S}\to\delta\text{-}\mathrm{Sch}_S,\quad T\mapsto(T,0)$$

corresponds, through the equivalence 4.5, to the functor

$$\eta^*:\mathrm{Sch}_{/S}\to(\mathrm{Sch}_{/S})^{\mathbb{D}(S)}.$$

delta-P **Definition 4.7.** Let

$$\mathscr{P}:\mathrm{Sch}_{/S}^{\mathrm{op}}\to\mathbf{Cat}$$

be a pseudofunctor on the category of S-schemes.

We define a pseudofunctor

$$\delta\text{-}\mathscr{P}:\delta\text{-}\mathrm{Sch}_S^{\mathrm{op}}\to\mathbf{Cat},\quad \delta\text{-}\mathscr{P}(X,\delta_X)=\mathscr{P}^{\mathbb{X}},$$

where $\mathbb{X}\in\mathbf{PreCat}(\mathrm{Sch}_{/S})$ is the $\mathbb{D}(S)$-action corresponding to X, considered as a precategory by 2.7.

diff-self *Remark* 4.8. Assume that the indexed data $\mathscr{P}:\mathrm{Sch}_{/S}^{\mathrm{op}}\to\mathbf{Cat}$ is a full sub-pseudofunctor of the self-indexing of $\mathrm{Sch}_{/S}$, i.e., for $Y\in\mathrm{Sch}_{/S}$, $\mathscr{P}(Y)$ is a full subcategory of $\mathrm{Sch}_{/Y}$. We can think of $\mathscr{P}$ as being associated to a class of morphisms in $\mathrm{Sch}_{/S}$ that is stable under pullback, Then, the indexed data δ-$\mathscr{P}$ is a natural differential analogue of that class in the sense that, for $(X,\delta_X)\in\delta\text{-}\mathrm{Sch}_S$,

$$\delta\text{-}\mathscr{P}(X,\delta_X)=\{(P,\delta_P)\to(X,\delta_X)\in\delta\text{-}\mathrm{Sch}_S:P\to X\in\mathscr{P}(X)\},$$

i.e., it consists of those morphisms of differential schemes with target (X,δ_X) whose underlying morphism of schemes belongs to the class $\mathscr{P}$.

Example 4.9. If $\mathscr{P}(Y)$ is the category of quasi-projective morphisms $Q\to Y$, then, given an S-differential scheme (X,δ_S), the fibre δ-$\mathscr{P}(X,\delta_X)$ is the category of S-differential scheme morphisms $(P,\delta_P)\to(X,\delta_X)$ such that $P\to X$ is quasi-projective.

4.10. **Differential schemes as formal group actions.** The *additive formal group scheme* is given as the formal scheme

$$\widehat{\mathbb{G}}_a = \mathrm{Spf}(\mathbb{Z}[[t]]),$$

where the group operation is deduced from comultiplication

$$\mathbb{Z}[[t]] \to \mathbb{Z}[[t]]\widehat{\otimes}\mathbb{Z}[[u]] \simeq \mathbb{Z}[[t,u]], \quad t \mapsto t+u,$$

and the identity section from the counit map

$$\mathbb{Z}[[t]] \to \mathbb{Z}, \quad t \mapsto 0.$$

The associated functor of points sends a ring R to the additive group of its nilpotents

$$\widehat{\mathbb{G}}_a(R) = \mathrm{Nil}(R).$$

diffsch-Ga-action **Fact 4.11** (Bardavid, bardavid [3, 8.3.1]). *There is bijective correspondence between formal group actions of $\widehat{\mathbb{G}}_a$ on a scheme X and systems of Hasse-Schmidt derivations on its structure sheaf $\mathscr{O}_X$.*

In particular, differential schemes in characteristic 0 *are precisely $\widehat{\mathbb{G}}_a$-actions,*

$$\delta\text{-}\mathrm{Sch}_{\mathbb{Q}} = (\mathrm{Sch}_{/\mathbb{Q}})^{\widehat{\mathbb{G}}_a}.$$

Indeed, an action is a formal scheme morphism

$$\rho : \widehat{\mathbb{G}}_a \times X \to X$$

satisfying the usual axioms. It is 'infinitesimal' in the sense that it does not affect the underlying topological space of X, and, at the level of structure sheaves, for U open in X, it is given by a 'Taylor expansion' map

$$\mathscr{O}_X(U) \to \mathscr{O}(U)[[t]], \quad f \mapsto \sum_i d_i(f)t^i,$$

where $(d_i)_{i\in\mathbb{N}}$ is a system of Hasse-Schmidt derivations.

In characteristic 0, we have that $\delta_i = \frac{\delta^i}{i!}$ for a derivation δ on $\mathscr{O}_X$. In terms of functors of points, the action is given by the expression

$$\mathrm{Nil}(R) \times \mathrm{Hom}(\mathrm{Spec}(R), X) \longrightarrow \mathrm{Hom}(\mathrm{Spec}(R), X), \tag{1}$$

$$(\epsilon, (\varphi, \varphi^\sharp)) \longmapsto (\varphi, \sum_i \frac{\varphi^\sharp \circ \delta^i}{i!}\epsilon^i), \tag{2}$$

where $\varphi^\sharp : \mathscr{O}_X \to \varphi_*\mathscr{O}_{\mathrm{Spec}(R)}$, and we consider $\epsilon \in \mathrm{Nil}(R)$ as a global section of $\mathscr{O}_{\mathrm{Spec}(R)}$, and the sum is finite because ϵ is nilpotent.

4.12. **Trajectories and leaves.** Let (X, δ_X) be an S-differential scheme, where S is a $\mathbb{Q}$-scheme. Let us write $G = \widehat{\mathbb{G}}_a \times S$ for the formal additive group scheme considered over S, let $\rho : G \times_S X \to X$ be the corresponding infinitesimal action by δ_X, and write $\psi = (\rho, p_2) : G \times_S X \to X \times_S X$.

Given a generalised point $x : T \to X$, the corresponding *action map* a_x is obtained as the pullback

$$\begin{array}{ccc} G \times_S T & \longrightarrow & G \times_S X \\ {\scriptstyle a_x}\downarrow & & \downarrow{\scriptstyle \psi} \\ X \times_S T & \xrightarrow{\mathrm{id}\times x} & X \times_S X \end{array}$$

and it is given more explicitly by $a_x = (\rho \circ \mathrm{id}_G \times x, p_2)$.

orbit **Definition 4.13.** With above notation, the *orbit of* x is the scheme-theoretic image of the action map a_x,

$$O(x) = \mathrm{Im}(a_x),$$

and the *reduced orbit of* x is the underlying reduced closed subscheme

$$|O|(x) = |O(x)|.$$

lemma-traj **Lemma 4.14.** *Let $x \in X$ be a scheme-theoretic point, and consider the corresponding morphism $\bar{x} : \mathrm{Spec}(\kappa(x)) \to X$. The scheme-theoretic image of the composite morphism*

$$t_x : G \times_S \mathrm{Spec}(\kappa(x)) \xrightarrow{a_x} X \times_S spec(\kappa(x)) \xrightarrow{\pi_1} X$$

is integral, and its generic point is a leaf.

Proof. The statement is local on X, so we may assume that $X = \mathrm{Spec}(A, \delta)$ is affine. The morphism a_x corresponds to the map

$$A \to \kappa(x)[[t]], \quad f \mapsto \sum_i \frac{\alpha(\delta^i f)}{i!} t^i,$$

where $\alpha : A \to \kappa(x)$ is associated with $\bar{x}$ and the point x corresponds to $\mathfrak{p} = \ker(\alpha)$. The kernel of the above map is

$$\mathfrak{p}_\sharp = \{f \in A : \delta^n(f) \in \mathfrak{p} \text{ for all } n\},$$

and Keigher has shown that $\mathfrak{p}_\sharp$ is prime in keigher-prime-diff-idl [?, 1.5]. ☐

trajectory **Definition 4.15.** If $x \in X$ is a scheme-theoretic point, we define its *trajectory* as the unique leaf $\mathrm{Traj}(x)$ satisfying

$$\mathrm{Im}(t_x) = \overline{\{\mathrm{Traj}(x)\}},$$

and 4.14 shows that the definition is meaningful and agrees with the notion from bardavid [3].

O-vs-Traj **Lemma 4.16.** *With the above notation, for $x \in X$, $\mathrm{Traj}(x)$ is the generic point of the scheme-theoretic image of $O(x)$ via π_1, i.e.*

$$\mathrm{Im}(t_x) = \pi_1(O(x)).$$

O-under-maps **Lemma 4.17.** *Let $f : X \to Y$ be a morphism of S-differential schemes. If $x : T \to X$ is a generalised point, and $y = f \circ x$ its image in Y, then $O(y)$ is the scheme-theoretic image of $O(x)$ under the morphism $f \times \mathrm{id}_T$,*

$$O(y) = (f \times \mathrm{id}_T)(O(x)).$$

Consequently, if $x \in X$ is a scheme theoretic point and $y = f(x)$, then

$$\mathrm{Traj}(y) = f(\mathrm{Traj}(x)).$$

Both lemmas are immediate using the familiar behaviour of scheme-theoretic images of the composite morphism.

O-basech **Lemma 4.18.** *With the above notation, let $\eta' : X \to S'$ be the base change of η by a morphism $S' \to S$. Let $x' : T \to X'$ be a generalised point of X', and let $x : T \to X$ be its projection. Then*

$$O(x') \simeq O(x).$$

Proof. By using the definition of orbits maps, we obtain a cartesian diagram

$$\begin{array}{ccc} G' \times_{S'} T & \longrightarrow & G \times_S T \\ {\scriptstyle a_{x'}} \downarrow & & \downarrow {\scriptstyle a_x} \\ X' \times_{S'} T & \longrightarrow & X \times_S T \end{array}$$

where the horizontal arrows are isomorphisms, so we conclude that scheme-theoretic images of the vertical arrows are isomorphic. □

O-precomp

Lemma 4.19. *Consider the composite* $x' : T' \xrightarrow{t} T \xrightarrow{x} X$*, where* t *is a free affine morphism. Then*

$$O(x') \simeq O(x) \times_{X \times_S T} X \times_S T'.$$

In particular, if $x \in X$ *is a scheme-theoretic point, and* $t : \mathrm{Spec}(L) \to \mathrm{Spec}(\kappa(x))$ *for some field* L *containing* $\kappa(x)$*, then*

$$\pi_1(|O|(x')) = \mathrm{Traj}(x).$$

Proof. Both squares of the diagram

$$\begin{array}{ccccc} G \times_S T' & \longrightarrow & G \times_S T & \longrightarrow & G \times_S X \\ \downarrow {\scriptstyle a_{x'}} & & \downarrow {\scriptstyle a_x} & & \downarrow {\scriptstyle \psi} \\ X \times_S T' & \xrightarrow{\mathrm{id} \times t} & X \times_S T & \xrightarrow{\mathrm{id} \times x} & X \times_S X \end{array}$$

are cartesian, which means that $a_{x'}$ is the base change of a_x by the affine free morphism $\mathrm{id} \times T$. The free base change formula for scheme-theoretic images holds even in the case where morphisms are not quasi-compact by [16, Prop. 4].

The second claim follows by taking scheme-theoretic images of the closed subschemes

$$\begin{array}{ccc} O(x') & \longrightarrow & O(x) \\ \downarrow & & \downarrow \\ X \times_S T' & \longrightarrow & X \times_S T \end{array}$$

via first projections into X. □

4.20. **Categorical scheme of leaves.** The following definition is inspired by the notion of 'espace des feuilles grossier' that appears in Bardavid's thesis [3, 4.2] and that of 'quotient discret catégorique' from Ayoub's paper [?, 3.2.4], which ultimately stems from the notion of *categorial quotient* familiar in algebraic geometry, see [28, 0.5], for example.

def-cat-schl

Definition 4.21. Let (X, δ_X) be an S-differential scheme. A scheme $T \in \mathrm{Sch}_{/S}$, together with a morphism of S-differential schemes $\eta : (X, \delta_X) \to C(T) = (T, 0)$ is a *categorical* S*-scheme of leaves*, if it is a universal morphism from (X, δ_X) to the functor C in the sense that any other morphism of S-differential schemes $\eta' : (X, \delta_X) \to C(T') = (T', 0)$ factors through η, i.e., there is a unique S-morphism

$f : T \to T'$ such that $\eta' = C(f) \circ \eta$. In other words, the solid part of the diagram

$$\begin{array}{ccc} (X, \delta_X) & \xrightarrow{\eta} & C(T) \\ & {\scriptstyle \eta'} \searrow \quad \swarrow {\scriptstyle \exists ! C(f)} & \\ & C(T') & \end{array}$$

can be uniquely completed by a dashed arrow to a commutative diagram.

no-pi0-affine

Remark 4.22. The categorical scheme of leaves need not exist for an arbitrary differential scheme (X, δ_X), but when it does, it is unique up to unique isomorphism, and we denote it

$$\pi_0(X).$$

Remark 4.23. Given a differential ring (A, δ_A), it is difficult to speculate whether $\pi_0(\mathrm{Spec}(A, \delta_A))$ exists. Note that the scheme $\mathrm{Spec}(\mathrm{Const}(A, \delta_A))$ only satisfies the universal property required of a categorical scheme of leaves in the category of *affine* differential schemes, but not necessarily in the category of all differential schemes.

cat-schl-pi0

Lemma 4.24. *Let $\mathbb{X} \in \mathbf{PreCat}(\mathrm{Sch}_{/S})$ be the $\mathbb{D}(S)$-action associated to an S-differential scheme (X, δ), considered as a precategory. Then, the categorical scheme of leaves of (X, δ_X) is isomorphic to the scheme of connected components*

$$\pi_0(\mathbb{X}) = \mathrm{Coeq}\,(X_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} X_0)$$

of $\mathbb{X}$ in the category $\mathrm{Sch}_{/S}$, whenever either of the objects exist. In other words,

$$\pi_0(\mathbb{X}) \simeq \pi_0(X).$$

Proof. Using 2.11, we obtain that

$$\delta\text{-}\mathrm{Sch}_S((X, \delta_X), C(T')) \simeq (\mathrm{Sch}_{/S})^{\mathbb{D}(S)}(\mathbb{X}, \eta^* T') \simeq \mathrm{Sch}_{/S}(\pi_0(\mathbb{X}), T'),$$

whenever $\pi_0(\mathbb{X})$ exists, hence $\pi_0(\mathbb{X})$ satisfies the universal property of the categorical scheme of leaves. □

4.25. **Geometric scheme of leaves.**

Definition 4.26. A morphism of differential schemes

$$\eta : (X, \delta) \to C(Q) = (Q, 0)$$

is called a *geometric quotient*, making Q a *geometric scheme of leaves*, if

(1) the *topological condition*: η is surjective and submersive;
(2) the *orbit condition*: for $x, x' \in X$,

$\eta(x) = \eta(x')$ implies that

there exists a field L extending $\kappa(x), \kappa(x')$ with $|O|(x_L) = |O|(x'_L)$,

where we write x_L for the composite $\mathrm{Spec}(L) \to \mathrm{Spec}(\kappa(x)) \xrightarrow{x} X$;
(3) the *sheaf condition*: the sequence of quasicoherent $\mathscr{O}_Q$-modules

$$0 \to \mathscr{O}_Q \xrightarrow{\eta^\sharp} \eta_* \mathscr{O}_X \xrightarrow{\eta_* \delta_X} \eta_* \mathscr{O}_X$$

is exact, i.e., $\mathscr{O}_Q \simeq \mathrm{Const}(\eta_* \mathscr{O}_X)$.

A morphism η as above is a *Bardavid quotient* (bardavid [3, 8.4.2]), when we replace the orbit condition by

(2') the *trajectory condition*: for $x, x' \in X$,

$$\eta(x) = \eta(x') \text{ implies } \mathrm{Traj}(x) = \mathrm{Traj}(x').$$

We note that variants of this notion appear in ayoub [?, 3.2.4].

bardq-catq **Fact 4.27** (bardavid [3, 8.4.3]). *A Bardavid quotient is a categorical quotient.*

geom-categ **Proposition 4.28.** *If $\eta(X, \delta) \to (Q, 0)$ is a geometric quotient, it is a Bardavid quotient. Hence, a geometric scheme of leaves is a categorical scheme of leaves.*

Proof. If L is a field extending $\kappa(x)$ and $\kappa(x')$ such that $|O|(x_L) = |O|(x'_L)$, then, using 4.19, we obtain

$$\mathrm{Traj}(x) = \pi_1(|O|(x_L)) = \pi_1(|O|(x'_L)) = \mathrm{Traj}(x'),$$

hence the orbit condition implies the trajectory condition. The second statement follows from 4.27. □

univ-quot **Definition 4.29.** A categorical/Bardavid/geometric quotient $\eta : (X, \delta) \to (Q, 0)$ is *universal*, if it remains such after an arbitrary base change $q : Q' \to Q$. A quotient is *uniform*, if it is stable under any flat base change q.

4.30. **Simple differential schemes.**

def-simplicity **Definition 4.31.** An S-differential scheme (X, δ_X) is *simple* with respect to the pseudofunctor $\mathscr{P} : \mathrm{Sch}_{/S} \to \mathbf{Cat}$, if its categorical scheme of leaves $\pi_0(X)$ exists and the coequaliser

$$X_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} X_0 \xrightarrow{\eta} \pi_0(X)$$

is universal for $\mathscr{P}$ in the sense of 2.14, i.e., if $\mathbb{X}$ has a $\mathscr{P}$-universal scheme of connected components, or, if η is a categorical quotient universal with respect to $\mathscr{P}$.

CX-P **Lemma 4.32.** *If an S-differential scheme (X, δ_X) is simple for $\mathscr{P}$ with categorical scheme of leaves $\eta_X : X \to \pi_0(X)$, the canonical functor*

$$C_X : \mathscr{P}(\pi_0(X)) \simeq \mathscr{P}^{\mathbf{\Delta}(\pi_0(\mathbb{X}))} \xrightarrow{\eta_{\mathbb{X}}^*} \mathscr{P}^{\mathbb{X}} \simeq \delta\text{-}\mathscr{P}(X, \delta_X)$$

is fully faithful, where we wrote $\eta_{\mathbb{X}} : \mathbb{X} \to \mathbf{\Delta}(\pi_0(\mathbb{X}))$ for the associated morphism of precategories inducing the pullback of precategory actions $\eta_{\mathbb{X}}^*$ as in 2.11.

Proof. Using 2.14, we have that the arrow in the above diagram is fully faithful, so the composite is too. □

Remark 4.33. When $\mathscr{P}$ is a sub-pseudocategory of the self-indexing of $\mathrm{Sch}_{/S}$ as in 4.8, we obtain that

$$C_X(Q \to \pi_0(X)) = (X, \delta_X) \times_{C(\pi_0(X))} C(Q),$$

so C_X agrees with the functor introduced in the classical Galois context 3.1.

qc-tor-trick **Lemma 4.34.** *Let*

$$0 \to \mathscr{F}_1 \xrightarrow{\alpha} \mathscr{F}_2 \xrightarrow{\beta} \mathscr{F}_3 \xrightarrow{\gamma} \mathscr{F}_4 \to 0$$

an exact sequence of quasicoherent modules on a scheme Y, such that

(1) α is universally injective, and
(2) $\mathscr{F}_4$ is flat.

Then, for any morphism $g : Y' \to Y$, the sequence

$$0 \to g^*\mathscr{F}_1 \xrightarrow{g^*\alpha} g^*\mathscr{F}_2 \xrightarrow{g^*\beta} g^*\mathscr{F}_3 \xrightarrow{g^*\gamma} g^*\mathscr{F}_4 \to 0$$

is exact.

Proof. Since all the assumptions and the claim are local in the Zariski topology, we may assume that Y and Y' are affine, say $Y = \mathrm{Spec}(A)$, $Y' = \mathrm{Spec}(A')$.

Consider the short exact sequences of A-modules

$$0 \to F_1 \xrightarrow{\alpha} F_2 \to \mathrm{im}(\beta) \to 0 \quad \text{and} \quad 0 \to \mathrm{im}(\beta) \to F_3 \xrightarrow{\gamma} F_4 \to 0.$$

Since α is universally injective, the first sequence is universally exact, so

$$0 \to g^*F_1 \xrightarrow{g^*\alpha} g^*F_2 \to g^*\,\mathrm{im}(\beta) \to 0$$

is exact, where $g^*F = A' \otimes_A F$. Applying g^* to the second sequence yields a long exact sequence for Tor,

$$\cdots \to \mathrm{Tor}_1^A(F_4, A') \to g^*\,\mathrm{im}(\beta) \to g^*F_3 \xrightarrow{g^*\gamma} g^*\mathscr{F}_4 \to 0.$$

Since F_4 is a flat A-module, we have that $\mathrm{Tor}_1^A(F_4, A') = 0$, so we can splice the two short exact sequence into the required one. □

univ-sh-cond

Lemma 4.35. *Let $\eta : (X, \delta) \to (Q, 0)$ be a morphism of S-differential schemes, and let $\eta' : X' \to Q'$ be the base change of η by a scheme morphism $q : Q' \to Q$. Suppose that either*

(1) η is an affine pure scheme morphism, and $\mathrm{coker}(\eta_*\delta)$ is a flat module on Q, or
(2) η is qcqs, and q is flat.

If η satisfies the sheaf condition for geometric quotients, so does η'.

Proof. If η satisfies the sheaf condition, we have an exact sequence of quasicoherent modules on Q,

$$0 \to \mathscr{O}_Q \to \eta_*\mathscr{O}_X \xrightarrow{\eta_*\delta} \eta_*\mathscr{O}_X \to \mathrm{coker}\,\eta_*\delta \to 0.$$

If q is flat, applying q^* yields an exact sequence. The flat base change formula [30, 02KE] gives that $q^*\eta_*\mathscr{O}_X \simeq \eta'_*\mathscr{O}_{X'}$, so we obtain that η' also satisfies the sheaf condition. Hence, (2) is proved. stacks-project

For (1), note that purity of η implies that $\eta^\sharp : \mathscr{O}_Q \to \eta_*\mathscr{O}_X$ is universally injective. Indeed, for an arbitrary q as above, purity tell us that the composite $q^*\mathscr{O}_Q \to q^*\eta_*\mathscr{O}_X \to \eta'_*\mathscr{O}_{X'}$ is injective, where the last morphism is the base change morphism. It follows that the first morphism is also injective.

Applying q^* to the above yields an exact sequence by 4.34. Using the assumption that η is affine, the affine base change stacks-project [30, 02KE] ensures $q^*\eta_*\mathscr{O}_X \simeq \eta'_*\mathscr{O}_{X'}$, so we obtain that η' satisfies the sheaf condition. □

geom-universal

Proposition 4.36. *Let $\eta : (X, \delta) \to (Q, 0)$ be an fpqc geometric quotient.*

(1) If η is affine and $\mathrm{coker}(\eta_*\mathscr{O}_X \xrightarrow{\eta_*\delta} \eta_*\mathscr{O}_X)$ is flat, then η is a universal geometric quotient and X is simple with respect to scheme morphisms.
(2) It is a uniform geometric quotient and X is simple with respect to flat scheme morphisms.
(3) If Q is the spectrum of a field, then η is a universal geometric quotient and X is simple with respect to scheme morphisms.

Proof. Since η is fpqc, it is universally submersive and surjective, so the topological condition is stable under arbitrary base change. Using 4.18 and 4.19, we see that the orbit condition is stable under arbitrary base change.

If η is faithfully flat, it is pure, so the stability of the sheaf condition in cases (1) and (2) follows from the corresponding statements of 4.35.

If Q is the spectrum of a field, then any morphism q with codomain Q is flat, so (3) is a special case of (2). □

affine-CX-remark

Remark 4.37. The affine version of 4.35 reads as follows.

Let (A, δ) be a differential ring with the ring of constants k, and R be a k-algebra such that either

(1) $k \to A$ is *universally injective* stacks-project [30, Tag 0581] and $\mathrm{coker}(\delta)$ is a flat k-module, or
(2) R is flat over k.

Then

$$\mathrm{Const}((A, \delta) \otimes_{(k,0)} (R, 0)) \simeq R.$$

This key condition is familiar from classical literature on differential Galois theory.

pushfwd-const-sh

Lemma 4.38. *If a differential scheme (X, δ_X) is simple with respect to Zariski open immersions, with categorical scheme of leaves given as coequaliser from 4.31, then it satisfies the sheaf condition for quotients,*

$$\eta_* \,\mathrm{Const}(\mathscr{O}_X, \delta_X) = \mathscr{O}_{\pi_0(X)}.$$

Proof. From the coequaliser diagram of schemes, we obtain a diagram of structure sheaves

$$\eta_*\mathscr{O}_{X_1} \underset{\eta_*\delta_1}{\overset{\eta_*\delta_0}{\leftleftarrows}} \eta_*\mathscr{O}_{X_0} \longleftarrow \mathscr{O}_{\pi_0(X)}$$

which yields a unique morphism

$$\mathscr{O}_{\pi_0(X)} \to \mathrm{Eq}(\eta_*\delta_0, \eta_*\delta_1) \simeq \eta_* \,\mathrm{Eq}(\delta_0, \delta_1) = \eta_* \,\mathrm{Const}(\mathscr{O}_X, \delta_X).$$

By assumption, for every Zariski open $V \hookrightarrow \pi_0(X)$, the diagram

$$U_1 \rightrightarrows U \longrightarrow V$$

with $U = \eta^{-1}V$ and U_1 the pullback of V to X_1, remains a coequaliser. Substituting $V = \mathrm{Spec}(A)$ in the above equaliser of sheaves, we obtain a ring homomorphism

$$A \to \mathrm{Eq}(\mathscr{O}_U(U) \rightrightarrows \mathscr{O}_{U_1}(U)).$$

The universal property of V being coequaliser, applied to varying affine schemes, yields that in fact

$$\mathscr{O}_{\pi_0(X)}(V) = A \simeq \mathrm{Eq}(\mathscr{O}_U(U) \rightrightarrows \mathscr{O}_{U_1}(U) = \eta_* \,\mathrm{Const}(\mathscr{O}_X)(V),$$

for an arbitrary affine open V, whence we obtain the desired conclusion. □

our-proof

Proposition 4.39. *Let (X, δ_X) be an S-differential scheme with categorical scheme of leaves S such that:*

(1) the canonical morphism

$$\eta : X \to \pi_0(X) = S$$

is fpqc and universally open (for example, fppf);

(2) for every Zariski open U in X, we have

$$\pi_0(U) = \eta(U);$$

(3) the module $\operatorname{coker}(\eta_* \delta_X)$ *is flat over S.*

Then (X, δ_X) is simple with respect to S-scheme morphisms, i.e., η is a universal categorical quotient.

Proof. Writing $\mathbb{X}$ for the precategory in $\mathrm{Sch}_{/S}$ associated to (X, δ_X), our first assumption means that the diagram

$$X_1 \overset{d_0}{\underset{d_1}{\rightrightarrows}} X_0 \longrightarrow S$$

is a coequaliser. We need to show that it is universal for S-scheme morphisms, i.e., that the base change

$$\begin{array}{ccccc} X_1 \times_S Q & \rightrightarrows & X_0 \times_S Q & \longrightarrow & Q \\ \downarrow & & \downarrow & & \downarrow q \\ X_1 & \overset{d_0}{\underset{d_1}{\rightrightarrows}} & X_0 & \xrightarrow{\ \eta\ } & S \end{array}$$

along any S-scheme morphism $Q \to S$ remains a coequaliser. Indeed, let $f : X_0 \times_S Q \to Y$ be a morphism of S-schemes that coequalises the two arrows from $X_1 \times_S Q$. Every point $p \in X_0 \times_S Q$ has an open affine neighbourhood $U_p \times_{S_p} Q_p$, where U_p, S_p and Q_p are open affine with $\eta(U_p) \subseteq S_p$ and $q(Q_p) \subseteq S_p$ such that $f_p = f \restriction_{U_p \times_{S_p} Q_p}$ factors through an affine open subset of Y. Since η is open, we may assume that $\eta(U_p) = S_p$.

As a base-change of a surjective morphism $X_0 \to S$, the morphism $X_0 \times_S Q \to Q$ is surjective, so, since $U_p \times_{S_p} Q_p$ cover $X_0 \times_S Q$, we obtain that Q_p cover Q. The morphisms d_0 and d_1 are identites on the underlying topological spaces of X_1 and X_0, and let us write $U_{p,1} = U_p \times_S S_1$ for the preimages of U_p in X_1.

By assumption,

$$U_{p,1} \rightrightarrows U_p \longrightarrow S_p$$

is a coequaliser, and by 4.37, we obtain that

$$U_{p,1} \times_S Q_i \rightrightarrows U_p \times_{S_p} Q_p \xrightarrow{\eta_p} Q_p$$

is a coequaliser in $\mathrm{Aff}_{/S_p}$.

Since f_p factors through an affine open in Y, there exists a unique $h_p : Q_p \to Y$ such that

$$f_p = h_p \circ \eta_p.$$

Every point in $Q_p \cap Q_{p'}$ has an affine open neighbourhood W which is standard open in Q_p and $Q_{p'}$, so by the same affine argument applied to $U_p \times_S W$ and $U_{p'} \times_S W$, we obtain that $h_p \restriction_W = h_{p'} \restriction_W$.

Hence

$$h_p \restriction_{Q_p \cap Q_{p'}} = h_{p'} \restriction_{Q_p \cap Q_{p'}},$$

so the h_p can be glued into a morphism

$$h : Q \to Y$$

verifying the universal property of coequaliser for Q. □

our-proof-cor

Proposition 4.40. *Let k be a field of characteristic 0, let $S = \mathrm{Spec}(k)$, and let (X, δ_X) be an integral S-differential scheme whose only leaf is the generic point (i.e., it is simple in the sense of* bardavid [3, 4.1.1]*). Then (X, δ_X) is simple with respect to S-scheme morphisms.*

Proof. Bardavid has shown bardavid [3, 4.3.6, 4.3.2] that S is the categorical scheme of leaves of (X, δ_X) and that S remains the categorical scheme of leaves for any open subscheme of X, so we may apply 4.39.

Alternatively, it is straightforward to check that $\eta : X \to S$ is a geometric quotient, so universality follows from 4.36. □

This proposition shows simplicity both of classical Picard-Vessiot rings, and of torsors associated with strongly normal extensions.

4.41. **Polarised differential scheme morphisms.**

def-polarised-qproj

Definition 4.42. The category of *polarised quasi-projective morphisms* has objects

$$(P, \mathscr{L}_P) \xrightarrow{p} U,$$

consisting of a scheme morphism $p : P \to U$ and a p-relatively ample invertible $\mathscr{O}_P$-module $\mathscr{L}_P$.

A morphism (f, α) between polarised quasi-projective morphisms $(P, \mathscr{L}_P) \xrightarrow{p} U$ and $(Q, \mathscr{L}_Q) \xrightarrow{q} V$ is a commutative diagram

$$\begin{array}{ccc} (P, \mathscr{L}_P) & \xrightarrow{(f,\alpha)} & (Q, \mathscr{L}_Q) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ U & \longrightarrow & V \end{array}$$

consisting of a scheme morphism $f : P \to Q$ that makes the underlying scheme diagram commutative, together with an $\mathscr{O}_P$-module *isomorphism* $\alpha : f^*\mathscr{L}_Q \to \mathscr{L}_P$.

If (g, β) is another morphism from $(Q, \mathscr{L}_Q) \xrightarrow{q} V$ to $(R, \mathscr{L}_R) \xrightarrow{r} W$, the composite is computed as

$$(g, \beta) \circ (f, \alpha) = (g \circ f, \alpha \circ f^*\beta \circ \text{coherence}),$$

where, more precisely, the $\mathscr{O}_P$-module isomorphism is given as the composite

$$(g \circ f)^*\mathscr{L}_R \simeq f^*g^*\mathscr{L}_R \xrightarrow{f^*\beta} f^*\mathscr{L}_Q \xrightarrow{\alpha} \mathscr{L}_P.$$

nota-polarised

Notation 4.43. The category of polarised quasi-projective morphisms has a natural codomain fibration over the category of schemes. In this subsection, we write

$$\mathscr{P} : \mathrm{Sch}_{/S}^{\mathrm{op}} \to \mathbf{Cat}$$

for the associated pseudofunctor, i.e., $\mathscr{P}(V)$ is the category consisting of pairs

$$(Q \xrightarrow{q} V, \mathscr{L}),$$

where q is a morphism of finite type and $\mathscr{L}$ is an invertible q-ample $\mathscr{O}_Q$-sheaf.

We will write

$$\mathscr{S} = \mathrm{Self}(\mathrm{Sch}_{/S})$$

for the self-indexing of the category of S-schemes over itself, and

$$U : \mathscr{P} \to \mathscr{S}$$

for the natural forgetful functor.

def-pol-dif **Definition 4.44.** The category of *polarised quasi-projective differential scheme morphisms* is the fibered category

$$\delta\text{-}\mathscr{P}$$

obtained by construction 4.7.

descr-poldif **Lemma 4.45.** *For an S-differential scheme (X, δ_X), the category of polarised quasi-projective differential scheme morphisms with codomain (X, δ) is equivalent to the category of pairs*

$$((P, \delta_P) \xrightarrow{p} (X, \delta_X), (\mathscr{L}_P, \delta_{\mathscr{L}_P})),$$

where $p : (P, \delta_P) \to (X, \delta_X)$ is a morphism of S-differential schemes, and $(\mathscr{L}_P, \delta_{\mathscr{L}_P})$ is an invertible $(\mathscr{O}_P, \delta_P)$-module with $\mathscr{L}_P$ relatively ample with respect to the underlying scheme morphism $P \to X$.

Proof. An object $P \in \delta\text{-}\mathscr{P}(X, \delta) = \mathscr{P}^{\mathbb{X}}$ gives rise to a diagram

$$\begin{array}{ccccc}
(P_2, \mathscr{L}_{P_2}) & \rightrightarrows & (P_1, \mathscr{L}_{P_1}) & \leftrightarrows & (P_0, \mathscr{L}_{P_0}) \\
\downarrow & & \downarrow & & \downarrow \\
X_2 & \rightrightarrows & X_1 & \leftrightarrows & X_0 \\
\downarrow & & \downarrow & & \downarrow \\
S_2 & \rightrightarrows & S_1 & \leftrightarrows & S_0
\end{array}$$

where the bottom level is the precategory $\mathbb{D}(S)$ as in 4.5, and all the arrows in the top and middle levels are cartesian. Hence, forgetting the polarisations, we see that P is also an S-differential scheme, whose differential structure is given by an S_1-automorphism $P_1 \xrightarrow{f} P_1$, which is identity on the underlying space, and it is given by an automorphism $\varphi : \mathscr{O}_{P_0}[\epsilon]/(\epsilon^2) \to \mathscr{O}_{P_0}[\epsilon]/(\epsilon^2)$ associated to an S-derivation δ_P on $\mathscr{O}_P$ as in 4.5, and the action on invertible sheaves is given by an isomorphism $(f, \alpha) : (p_1, d^*_{P,0}\mathscr{L}_0) \to (p_1, d^*_{P,0}\mathscr{L}_0)$, i.e., by an isomorphism

$$\alpha : \mathscr{O}_{P_0}[\epsilon]/(\epsilon^2) \,{}_{\varphi}\!\otimes_{\mathscr{O}_{P_0}[\epsilon]/(\epsilon^2)} \mathscr{L}_0[\epsilon]/(\epsilon^2) \to \mathscr{L}_0[\epsilon]/(\epsilon^2).$$

The identity section requirement yields that

$$\alpha = \mathrm{id} + \epsilon\delta_\alpha \circ \eta_L,$$

where $\eta_L : \mathscr{L}_0[\epsilon]/(\epsilon^2) \to \mathscr{L}_0$ is the augmentation morphism, and $\delta_\alpha : \mathscr{L}_0 \to \mathscr{L}_0$ makes $\mathscr{L}_0$ into an $(\mathscr{O}_P, \delta_P)$-module. □

coeq-desc-inv-sh **Lemma 4.46.** *Let (X, δ_X) be a differential scheme which is simple with respect to Zariski open immersions, with the categorical scheme of leaves given by the coequaliser*

$$X_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} X_0 \xrightarrow{\eta} \pi_0(X).$$

Consider invertible sheaves $\mathscr{L}$ and $\mathscr{L}'$ on $\pi_0(X)$ and an $\mathscr{O}_{X_0}$-module isomorphism

$$\alpha : \eta^*\mathscr{L} \to \eta^*\mathscr{L}'$$

making the diagram

$$\begin{array}{ccc} d_0^*\eta^*\mathscr{L} & \xrightarrow{d_0^*\alpha} & d_0^*\eta^*\mathscr{L}' \\ \wr\downarrow & & \downarrow\wr \\ d_1^*\eta^*\mathscr{L} & \xrightarrow{d_1^*\alpha} & d_1^*\eta^*\mathscr{L}' \end{array}$$

commutative.

Then, there exists a unique isomorphism $\sigma : \mathscr{L} \to \mathscr{L}'$ such that

$$\alpha = \eta^*\sigma.$$

Proof. Let V_i constitute an open cover of $\pi_0(X)$ such that, for every i, $\mathscr{L} \restriction_{V_i}$ and $\mathscr{L}' \restriction_{V_i}$ are free of rank 1. Then $U_i = \eta^{-1}(V_i)$ cover X_0 and $\eta^*\mathscr{L} \restriction_{U_i} \simeq \mathscr{O}_{U_i} e_i$, $\eta^*\mathscr{L}' \restriction_{U_i} \simeq \mathscr{O}_{U_i} e_i'$ are free rank 1, for some e_i and e_i', so $\alpha \restriction_{U_i}$ is given as multiplication by some $a_i \in \mathscr{O}_X(U_i)^\times$.

By identifying

$$d_0^*\eta^*\mathscr{L} \restriction_{U_i} = \mathscr{O}_{U_i}[\epsilon]/(\epsilon^2) \otimes_{\mathscr{O}_{U_i}} \mathscr{O}_{U_i} e_i \simeq \mathscr{O}_{U_i}(1 \otimes e_i) \oplus \mathscr{O}_{U_i}(\epsilon \otimes e_i),$$

and similarly for $d_0^*\eta^*\mathscr{L}' \restriction_{U_i}$, the matrix of $d_0^*\alpha \restriction_{U_i}$ in the pair of bases $((1 \otimes e_i), (\epsilon \otimes e_i))$, $((1 \otimes e_i'), (\epsilon \otimes e_i'))$ becomes

$$\begin{pmatrix} a_i & 0 \\ 0 & a_i \end{pmatrix}.$$

In the same pair of bases, the matrix of $d_1^*\alpha \restriction_{U_i}$ is

$$\begin{pmatrix} a_i & 0 \\ \delta_X(a_i) & a_i \end{pmatrix}.$$

By the assumption that $d_0^*\alpha$ and $d_1^*\alpha$ agree up to coherences, we conclude that $\delta_X(a_i) = 0$, i.e., by 4.38,

$$a_i \in \operatorname{Const}(\mathscr{O}_X)(U_i) = \eta_* \operatorname{Const}(\mathscr{O}_X)(V_i) \simeq \mathscr{O}_{\pi_0(X)}(V_i),$$

whence it gives an isomorphism $\mathscr{L} \restriction_{V_i} \to \mathscr{L}' \restriction_{V_i}$.

By the same argument, these isomorphisms agree on the intersections $V_i \cap V_j$ and hence glue uniquely to an isomorphism $\mathscr{L} \to \mathscr{L}'$ with the desired property. □

reflect-quasiproj-coeq

Proposition 4.47. *The forgetful functor*

$$\delta\text{-}\mathscr{P} \to \delta\text{-}\mathscr{S}$$

reflects coequalisers associated with connected components of differential schemes which are simple with respect to Zariski open immersions.

Proof. Let $(X, \delta_X) \in \delta\text{-}\mathscr{S}$ be simple for Zariski open immersions, and let $(p_2, p_1, p_0) \in \delta\text{-}\mathscr{P}(X, \delta_X)$ be such that, in the diagram

$$\begin{array}{ccccc} (P_1, \mathscr{L}_{P_1}) & \overset{(d_0,\delta_0)}{\underset{(d_1,\delta_1)}{\rightrightarrows}} & (P_0, \mathscr{L}_{P_0}) & \xrightarrow{(r,\rho)} & (Q, \mathscr{L}_Q) \\ {\scriptstyle p_1}\downarrow & & {\scriptstyle p_0}\downarrow & & \downarrow \\ X_1 & \rightrightarrows & X_0 & \longrightarrow & \pi_0(X) \end{array}$$

we have that (r, ρ) coequalises (d_0, δ_0) and (d_1, δ_1), and the underlying diagram of scheme morphisms is a coequaliser.

We need to show that the diagram is a coequaliser of polarised quasi-projective morphisms. Let $(h, \chi) : (P_0, \mathscr{L}_{P_0}) \to (T, \mathscr{L}_T)$ be an arbitrary morphism that coequalises (d_0, δ_0) and (d_1, δ_1). Since the underlying diagram of scheme morphisms is a coequaliser, there is a unique morphism $s : Q \to T$ such that $s \circ r = h$.

Let $\mathscr{L} = s^*\mathscr{L}_T$, $\mathscr{L}' = \mathscr{L}_Q$, and consider the isomorphism $\alpha : r^*\mathscr{L} \to r^*\mathscr{L}'$ given as the composite

$$r^*s^*\mathscr{L}_T \simeq h^*\mathscr{L}_T \xrightarrow{\chi} \mathscr{L}_{P_0} \xrightarrow{\rho^{-1}} r^*\mathscr{L}_Q.$$

Conditions $(h, \chi) \circ (d_0, \delta_1) = (h, \chi) \circ (d_0, \delta_1)$ and $(r, \rho) \circ (d_0, \delta_1) = (r, \rho) \circ (d_0, \delta_1)$ show that the diagram

$$\begin{array}{ccccc} d_0^*h^*\mathscr{L}_T & \xrightarrow{d_0^*\chi} & d_0^*\mathscr{L}_{P_0} & \xrightarrow{d_0^*\rho^{-1}} & d_0^*r^*\mathscr{L}_Q \\ \Big\downarrow \wr & & \delta_0 \Big\downarrow & & \Big\downarrow \wr \\ & & \mathscr{L}_{P_1} & & \\ & & \delta_1 \Big\uparrow & & \\ d_1^*h^*\mathscr{L}_T & \xrightarrow{d_1^*\chi} & d_0^*\mathscr{L}_{P_0} & \xrightarrow{d_1^*\rho^{-1}} & d_1^*r^*\mathscr{L}_Q \end{array}$$

commutes, so $d_0^*\alpha$ and $d_1^*\alpha$ agree up to coherences. By 4.46, we obtain an isomorphism

$$\sigma : s^*\mathscr{L}_T = \mathscr{L} \to \mathscr{L}' = \mathscr{L}_Q$$

such that $\alpha = r^*\sigma$, and it follows that

$$(s, \sigma) \circ (r, \rho) = (h, \chi),$$

so $(Q, \mathscr{L}_Q)$ has the universal property of a coequaliser, as required. □

4.48. **Differential descent.**

diffl-desc **Proposition 4.49.** *Let $\mathscr{P} : \mathrm{Sch}_{/S}^{\mathrm{op}} \to \mathbf{Cat}$ be a pseudofunctor, and let*

$$f : (X, \delta_X) \to (Y, \delta_Y)$$

be a morphism of S-differential schemes such that

1. *the underlying morphism $f_0 : X \to Y$ is a morphism of effective descent for $\mathscr{P}$;*
2. *$f_1 = d_0^* f_0 : X_1 \to Y_1$ is a descent morphism for $\mathscr{P}$, where $d_0 : S_1 \to S_0 = S$ is the source morphism of precategory $\mathbb{D}(S)$.*

Then f is a morphism of effective descent for δ-$\mathscr{P}$.

Proof. We consider the morphism of S-differential schemes as a morphism $f : \mathbb{X} \to \mathbb{Y}$ of $\mathbb{D}(S)$-actions, and apply 2.29, noting that the cocycle condition is superfluous for differential schemes so we may omit the condition on f_2. □

diffl-desc-qp **Corollary 4.50.** *Let $f : (X, \delta_X) \to (Y, \delta_Y)$ be a morphism of differential schemes with codomain the spectrum of a differential field and X quasi-compact. Then f is a morphism of effective descent for the class of differential quasi-projective morphisms.*

Proof. The underlying morphism $f_0 : X \to Y$ is (trivially) fpqc because the target is the spectrum of a field, and 2.31 show that it is effective descent for quasi-projective scheme morphisms. Its base change $f_1 = d_0^* f$ is again fpqc, hence a morphism of descent for all scheme morphisms by sga1 [15, VIII, 5.2]. □

difl-qproj-desc

Corollary 4.51. *Let* $f : (X, \delta_X) \to (Y, \delta_Y)$ *be a morphism of differential schemes whose underlying scheme morphism is fpqc. Then* f *is a morphism of effective descent for the class of differential polarised quasi-projective morphisms.*

Proof. The underlying scheme morphism $f_0 : X \to Y$ is fpqc, so of effective descent for polarised quasi-projective morphisms by [sga1] [15, VIII, 7.8], and the same holds for f_1 as the base change of f_0. □

pv-c[illegible]

5. Affine Picard-Vessiot theory

5.1. **Picard-Vessiot Galois theory for differential field extensions.** In this section, we follow the Hopf-theoretic approach to Picard-Vessiot theory explained in [amano-masuoka-takeuchi] [1].

Definition 5.2. An extension L/K of differential fields is *Picard-Vessiot*, if

(1) the extension L contains no new constants, i.e., $L_0 = K_0$, and we write k for the common field of constants;
(2) there exists a differential K-subalgebra A of L such that $\mathrm{Frac}(A) = L$ and
$$H = (A \otimes_K A)_0$$
generates the left A-module $A \otimes_K A$ in the sense that
$$(A \otimes_K K)H = A \otimes_K A.$$

Such an A is called a *Picard-Vessiot ring* for the extension L/K.

fact-pvring

Fact 5.3. *In the situation from the above definition, we have the following.*

(1) A *is unique;*
(2) H *is a Hopf algebra;*
(3) there is a comodule structure $\theta : A \to A \otimes_k H$ *such that*
$$A^{\mathrm{co}H} = \{a \in A : \theta(a) = a \otimes 1\} = K,$$
and

selfsplitting

$$(\dagger) \qquad A \otimes_K A \simeq A \otimes_{(k,0)} (H, 0).$$

(4) The linear algebraic group $G = \mathrm{Gal}^{\mathrm{PV}}(L/K) = \mathrm{Spec}(H)$ *over* k *is called the* Picard-Vessiot Galois group *of* L/K *and we have that*
$$G(k) \simeq \mathrm{Aut}_{\delta\text{-}\mathrm{Rng}}(L/K).$$

class-pv-corr

Fact 5.4 (Classical Picard-Vessiot Galois correspondence)**.** *Let* L/K *be a Picard-Vessiot extension.*

There is a one-to-one correspondence between intermediate differential field extensions and closed subgroups of the linear algebraic group $\mathrm{Gal}^{\mathrm{PV}}(L/K)$ *given by*
$$M \mapsto \mathrm{Gal}^{\mathrm{PV}}(L/M).$$

Moreover, an intermediate field M *in* L/K *is Picard-Vessiot over* K *if and only if* $\mathrm{Gal}^{\mathrm{PV}}(L/M)$ *is normal in* $\mathrm{Gal}^{\mathrm{PV}}(L/K)$*. In this case, we have that*
$$\mathrm{Gal}^{\mathrm{PV}}(M/K) \simeq \mathrm{Gal}^{\mathrm{PV}}(L/K)/\mathrm{Gal}^{\mathrm{PV}}(L/M).$$

pv-janelidze

5.5. Janelidze's categorical framework for Picard-Vessiot theory.

affine-pv

Definition 5.6. In [17], Janelidze makes the following choices for the classical categorical setup as in 3.1.

(1) $\mathscr{A} = \delta\text{-Aff} = \delta\text{-Rng}^{\mathrm{op}}$, the category of affine differential schemes;
(2) $\mathscr{X} = \mathrm{Aff} = \mathrm{Rng}^{\mathrm{op}}$, the category of affine schemes;
(3) $S = \mathrm{Const}^{\mathrm{op}}$ is the functor of constants, i.e., $S(\mathrm{Spec}(A,\delta)) = \mathrm{Spec}(\mathrm{Const}(A,\delta))$, often written as $S(X) = X_0$;
(4) $C(Spec(R)) = Spec(R, 0)$ transforms a ring into a differential ring with a trivial derivation 0.

Remark 5.7. (1) Given $X \in \mathscr{A}$, the functor $C_X : \mathscr{X}_{/X_0} \to \mathscr{A}_{/X}$ is given by

$$C_X(Q \to X_0) = X \times_{C(X_0)} C(Q) = (X, \delta) \times_{(X_0,0)} (Q, 0).$$

(2) An object $P \xrightarrow{p} Y$ in $\mathscr{A}_{/Y}$ is split by a morphism $X \xrightarrow{f} Y$ if the natural morphism $f^*p \to C_X S_X(f^*p)$ is an isomorphism, i.e., if

$$X \times_Y P \simeq X \times_{(X_0,0)} ((X \times_Y P)_0, 0).$$

janelidze-affine

Theorem 5.8 ([17]). *Let A be the Picard-Vessiot ring for a differential field extension L/K, and let*

$$f = \mathrm{Spec}(K \to A) : X = \mathrm{Spec}(A) \to Y = \mathrm{Spec}(K)$$

be the associated morphism in $\mathscr{A}$.

Then f is a morphism of Galois descent, the categorical Galois groupoid agrees with the Picard-Vessiot Galois group,

$$G = \mathrm{Gal}[f] = \mathrm{Gal}^{\mathrm{PV}}(L/K),$$

and X is a G-torsor over Y in the sense that

$$X \times_Y X \simeq X \times_{(X_0,0)} (G, 0).$$

There is an equivalence of categories

$$\mathrm{Split}_Y[f] \simeq [G, \mathscr{X}]$$

between the category of objects $P \xrightarrow{p} Y$ in $\mathscr{A}_{/Y}$ split by f, in the sense that

$$X \times_Y P = f^*(p) \simeq C_X(q) = X \times_{X_0} Q$$

for some $Q \xrightarrow{q} X_0$ in $\mathscr{X}_{/X_0}$, and the category of G-actions in $\mathscr{X}$.

Proof. (1) The morphism f is faithfully flat (given as a spectrum of an algebra over a field) and hence it is a morphism of effective descent for affine morphisms in the sense of algebraic geometry. Using Benabou-Roubaud [5], we obtain that the pullback functor f^* is monadic.
(2) Using 4.37, the counit $S_X C_X \to \mathrm{id}$ is an isomorphism, or, equivalently, C_X is fully faithful.
(3) The morphism f is self-split by the property † from 5.3.

Categorical Galois theory stipulates that the object of morphisms of the groupoid $\mathrm{Gal}[f]$ is

$$G = \mathrm{Gal}[f]_1 = S(f^*f) = (X \times_Y X)_0 = \mathrm{Spec}((A \otimes_K A)_0) = \mathrm{Spec}(H),$$

while the object of objects

$$\mathrm{Gal}[f]_0 = S(X) = X_0 = \mathrm{Spec}(k)$$

is a point, so we obtain a linear algebraic group G over k, exactly as in the Picard-Vessiot case. The torsor equation is precisely the self-splitting of f, written in terms of G, and the equivalence of categories follows from categorical Galois theory 3.1 specialised to the framework 5.6. □

effective-alg-subgp

Fact 5.9 (Algebraic group quotients and effective subgroups)**.** *Let $H \to G$ be a monomorphism/closed immersion of algebraic groups over a field k. Combining* milne [27, 5.24, 5.28, 8.42–8.44, B.37, B.38]*, or, by using* sga3.1 [12, Exposé V]*, we obtain:*

(1) G is quasi-projective;
(2) the quotient G/H is representable by a quasi-projective scheme over k;
(3) the quotient morphism $G \to G/H$ is faithfully flat;
(4) we have

$$G \times H \simeq G \times_{G/H} G.$$

We deduce that all closed subgroups of an algebraic group over a field k are effective in the category of schemes with quasi-projective morphisms.

*In the category of affine schemes, a closed subgroup H of an affine algebraic group G is effective if and only if G/H is affine. If H is a normal closed subgroup, then it is effective (*milne [27, 5.29]*).*

affine-pv-corr

Proposition 5.10 (Affine Picard-Vessiot correspondence)**.** *With assumptions of 5.8, there is a one-to-one correspondence between split affine quotients of $f : X \to Y$ and effective subgroups of the linear algebraic group $G = \mathrm{Gal}[f]$ which takes*

$$\begin{array}{ccc} & & X \\ & \swarrow & \Big\downarrow f \\ P & & \\ & \searrow_{p} & \\ & & Y \end{array}$$

to

$$\mathrm{Gal}[X \to P].$$

Conversely, if G' is an effective subgroup in the sense that it is a closed subgroup such that the coset space G/G' has a structure of an affine scheme (5.9), it corresponds to the quotient

$$X/G',$$

which is f-split by the scheme G/G'.

Moreover, this correspondence restricts to a one-to-one correspondence between split quotients P such that $P \to Y$ is Picard-Vessiot, and closed normal subgroups of G. In this case,

$$\mathrm{Gal}[P \to Y] \simeq \mathrm{Gal}[X \to Y]/\,\mathrm{Gal}[X \to P].$$

Proof. The statement is a direct consequence of 5.8 and 3.4. □

Remark 5.11. (1) The equivalence of categories form of Picard-Vessiot theory from 5.8 is new and as of yet unexplored in differential algebra.

(2) The affine Galois correspondence from 5.10 does not fully recover the classical Picard-Vessiot Galois correspondence 5.4 because it only refers to *effective* subgroups of the Galois group, while the classical correspondence is for all closed subgroups. The 'moreover' clause does recover the correspondence for Picard-Vessiot quotients and normal groups from maurischat [26, 8.1].

s:sat-gal-dif

6. Categorical Galois theory for differential schemes

6.1. Indexed framework for scheme-theoretic Picard-Vessiot theory.

sch-pv-setup

Definition 6.2. The indexed framework for *pre-Picard-Vessiot* differential Galois theory consists of the following choices for objects needed to apply 3.6.

(1) Let S be a base scheme, and let $\mathscr{X} = \mathrm{Sch}_{/S}$.
(2) Let $\mathscr{A}$ be the category of S-differential schemes that have a categorical scheme of leaves. By this choice, we have a 'categorical scheme of leaves' functor
$$\pi_0 : \mathscr{A} \to \mathscr{X}.$$
(3) Let $\mathscr{P} : \mathscr{X}^{\mathrm{op}} \to \mathbf{Cat}$ be a pseudofunctor, which yields a pseudofunctor $\delta\text{-}\mathscr{P} : \mathscr{A}^{\mathrm{op}} \to \mathbf{Cat}$ by 4.7.
(4) Let $C = C^{\mathscr{P}} : \mathscr{P} \circ \pi_0 \Rightarrow \delta\text{-}\mathscr{P}$ be the pseudo-natural transformation whose (Z, δ_Z)-component is the canonical functor
$$C_Z : \mathscr{P}(\pi_0(Z, \delta_Z)) \to \delta\text{-}\mathscr{P}(Z, \delta_Z)$$
from 4.32.

For *Picard-Vessiot* differential Galois theory, we choose the following additional structure.

(5) Let $\mathscr{S} : \mathscr{X}^{\mathrm{op}} \to \mathbf{Cat}$ be a full sub-pseudofunctor of the self-indexing $\mathrm{Self}(\mathrm{Sch}_{/S})$ of the category of S-schemes over itself, so that, for an S-scheme Z, $\mathscr{S}(Z)$ is a full subcategory of $\mathrm{Sch}_{/Z}$. It gives rise to pseudofunctor $\delta\text{-}\mathscr{S} : \mathscr{A}^{\mathrm{op}} \to \mathbf{Cat}$.
(6) Let $C^{\mathscr{S}} : \mathscr{S} \circ \pi_0 \Rightarrow \delta\text{-}\mathscr{P}$ be the pseudo-natural transformation corresponding to $\mathscr{S}$.
(7) Let $U : \mathscr{P} \Rightarrow \mathscr{S}$ be a faithful pseudo-natural transformation. It gives rise to a morphism of fibrations, taking cartesian morphisms to cartesian, hence we obtain a pseudo-natural transformation $\delta\text{-}U : \delta\text{-}\mathscr{P} \Rightarrow \delta\text{-}\mathscr{S}$ such that the diagram
$$\begin{array}{ccc} \mathscr{P} \circ \pi_0 & \xrightarrow{C^{\mathscr{P}}} & \delta\text{-}\mathscr{P} \\ {\scriptstyle U}\downarrow & & \downarrow{\scriptstyle \delta\text{-}U} \\ \mathscr{S} \circ \pi_0 & \xrightarrow{C^{\mathscr{S}}} & \delta\text{-}\mathscr{S} \end{array}$$
commutes.
(8) We require that $\delta\text{-}U$ reflects (coequalisers associated with) $\mathscr{S}$-universal connected components, i.e., if $P \in \delta\text{-}\mathscr{P}(X, \delta_X)$ for some $(X, \delta_X) \in \mathscr{A}$, and we have a diagram
$$P_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} P_0 \xrightarrow{r} Q.$$

with $r \circ d_0 = r \circ d_1$, which U maps onto an $\mathscr{S}$-universal coequaliser

$$\begin{array}{ccccc} UP_1 & \rightrightarrows & UP_0 & \longrightarrow & \pi_0(\delta\text{-}U(P)) \\ \downarrow & & \downarrow & & \downarrow \\ X_1 & \rightrightarrows & X_0 & \longrightarrow & \pi_0(X) \end{array}$$

associated to the $\mathscr{S}$-simple differential scheme δ-$U(P)$, then the original diagram was already a coequaliser.

Remark 6.3. In view of 4.22, the above setup with the functor π_0 and the pseudofunctor C is *not* an extension of the adjunction $()_0 \dashv C$ we used in 5.6, given that the functor $()_0$ on affine differential schemes does not extend to a functor on differential schemes.

Remark 6.4. The category

$$\mathrm{Split}_C(f)$$

consists of objects $P \in \delta$-$\mathscr{P}(Y)$ such that, for some $Q \in \mathscr{P}(S(X))$,

$$f^*P \simeq C_X(Q).$$

def-pre-pv

Definition 6.5. A morphism $f : (X, \delta_X) \to (Y, \delta_Y)$ of S-differential schemes is *pre-Picard-Vessiot* with respect to $U : \mathscr{P} \Rightarrow \mathscr{S}$ provided:

(0) f is a morphism in $\mathscr{A}$, i.e., (X, δ_X) and (Y, δ_Y) have categorical schemes of leaves over S;
(1) f is a morphism of effective descent for δ-$\mathscr{P}$;
(2) X, $X \times_Y X$ and $X \times_Y X \times_Y X$ are simple for $\mathscr{S}$.

We say that f is *pre-Picard-Vessiot* with respect to $\mathscr{P}$, if it is so with respect to $\mathrm{id} : \mathscr{P} \Rightarrow \mathscr{P}$.

def-pv

Definition 6.6. A morphism $f : (X, \delta_X) \to (Y, \delta_Y)$ of S-differential schemes is *Picard-Vessiot* with respect to $U : \mathscr{P} \Rightarrow \mathscr{S}$ provided:

(0) f is a morphism in $\mathscr{A}$, i.e., (X, δ_X) and (Y, δ_Y) have categorical schemes of leaves over S;
(1) f is a morphism of effective descent for δ-$\mathscr{P}$;
(2) X is simple for $\mathscr{S}$;
(3) f is auto-split with respect to $\mathscr{S}$, i.e., $f \in \mathrm{Split}_{C^{\mathscr{S}}}(f)$.

If $\mathscr{P}$ is already a sub-pseudofunctor of the self-indexing of $\mathrm{Sch}_{/S}$, we say that f is *Picard-Vessiot* with respect to $\mathscr{P}$, if it is such with respect to $\mathrm{id} : \mathscr{P} \Rightarrow \mathscr{P}$.

pv-for-P

Remark 6.7. A morphism $f : (X, \delta_X) \to (Y, \delta_Y)$ in $\mathscr{A}$ is *Picard-Vessiot* with respect to $\mathscr{P}$ provided:

(1) f is a morphism of effective descent for δ-$\mathscr{P}$;
(2) X is simple for $\mathscr{P}$;
(3) f is auto-split, i.e., $f \in \mathrm{Split}_C(f)$.

properties-pv

Lemma 6.8. *(1) If (X, δ_X) is simple for $\mathscr{S}$, then it is simple for $\mathscr{P}$.*
(2) Given an S-differential scheme (Y, δ_Y), the functor δ-U_Y restricts to

$$\mathrm{Split}_{C^{\mathscr{P}}}(f) \to \mathrm{Split}_{C^{\mathscr{S}}}(f).$$

Proof. For the first claim, suppose (X, δ_X) is simple with respect to $\mathscr{S}$, and let $Q \in \mathscr{P}(\pi_0(X))$. Let $P = C_X^{\mathscr{P}}(Q)$ and consider the associated diagram

$$P_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} P_0 \xrightarrow{r} Q.$$

Since (X, δ_X) is simple with respect to $\mathscr{S}$, the analogous diagram for δ-$U(C_X^{\mathscr{P}}(Q)) = C_X^{\mathscr{S}}(UQ)$ is an $\mathscr{S}$-universal coequaliser and $\pi_0(C_X^{\mathscr{S}}(UQ) = UQ$. Using the fact that δ-U reflects connected components, we deduce that the original diagram is a coequaliser and $\pi_0(P) = Q$.

The second claim follows directly from the fact that U preserves cartesian morphisms. □

PV-pre-PV

Lemma 6.9. *If f is a Picard-Vessiot morphism of differential schemes with respect to $U : \mathscr{P} \Rightarrow \mathscr{S}$, then f is pre-Picard-Vessiot with respect to U.*

Proof. Writing $G_0 = \pi_0(X)$, since f is auto-split with respect to $C^{\mathscr{S}}$, for some $G_1 \in \mathscr{S}(G_0)$,

$$X \times_Y X \simeq C_X^{\mathscr{S}}(G_1) = X \times_{C(G_0)} C(G_1).$$

By the definition of simplicity using universal coequalisers, any object in the essential image of $C_X^{\mathscr{S}}$ is automatically simple for $\mathscr{S}$, whence $X \times_Y X$ is simple for $\mathscr{S}$ and we have $\pi_0(X \times_Y X) \simeq G_1$. Moreover,

$$\begin{aligned} X \times_Y X \times_Y X &\simeq (X \times_Y X) \times_X (X \times_Y X) \\ &\simeq (C(G_1) \times_{C(G_0)} X) \times_X (X \times_{C(G_0)} C(G_1)) \simeq X \times_{C(X_0)} C(G_1 \times_{G_0} G_1) \\ &\simeq C_X^{\mathscr{S}}(G_1 \times_{G_0} G_1), \end{aligned}$$

whence $X \times_Y X \times_Y X$ is also simple for $\mathscr{S}$ with $\pi_0(X \times_Y X \times_Y X) \simeq G_1 \times_{G_0} G_1$, as required. □

galprecatdef

Remark 6.10. The assumption that f is pre-Picard-Vessiot for $\mathscr{P}$ ensures that the kernel-pair groupoid

$$\mathbb{G}_f : \qquad X \times_Y X \times_Y X \underset{\pi_{12}}{\overset{\pi_{01}}{\underset{\pi_{02}}{\rightrightarrows}}} X \times_Y X \overset{\pi_0}{\underset{\pi_1}{\underset{\Delta}{\leftleftarrows}}} X$$

is a category in $\mathscr{A}$, and that the *Galois precategory*

$$\mathrm{Gal}[f] = \pi_0(\mathbb{G}_f)$$

exists as a precategory in $\mathscr{X}$.

If f is Picard-Vessiot, then $G[f]$ is an internal *category* (actually a groupoid without the inversion of arrows named) in $\mathscr{X}$, by the argument in 6.9.

scheme-dif-Galois

Theorem 6.11. *A pre-Picard-Vessiot morphism f for $\mathscr{P}$ induces an equivalence of categories*

$$\mathrm{Split}_C(f) \simeq \mathscr{P}^{\mathrm{Gal}[f]}$$

between the category of objects of δ-$\mathscr{P}(Y)$ C-split by f and the category of $\mathscr{P}$-actions of the precategory $\mathrm{Gal}[f]$.

Moreover, if f is Picard-Vessiot for $U : \mathscr{P} \Rightarrow \mathscr{S}$, the latter becomes the category of $\mathscr{P}$-actions of the groupoid $\mathrm{Gal}[f]$.

Proof. When f is pre-Picard-Vessiot, 6.10 ensures that $\mathbb{G}_f$ gives a precategorical decomposition of f in $\mathscr{A}$, and that the Galois precategory $\mathrm{Gal}[f]$ is well-defined in $\mathscr{X}$. By assumption, f is of effective descent for $\mathbb{G}_f$.

By the assumption on simplicity of X, $X \times_Y X$, $X \times_Y X \times_Y X$ and 4.32, we get that the functors C_X, $C_{X \times_Y X}$, $C_{X \times_Y X \times_Y X}$ are fully faithful.

Hence, all the assumptions of 3.6 are satisfied. When f is Picard-Vessiot, it is also pre-Picard-Vessiot by 6.9.

Therefore, in both cases we get the desired equivalence involving actions of the precategory $\mathrm{Gal}[f]$, which happens to be a groupoid in the Picard-Vessiot case, as observed in 6.10. □

6.12. **Specialisation of the Galois precategory.**

special-gal

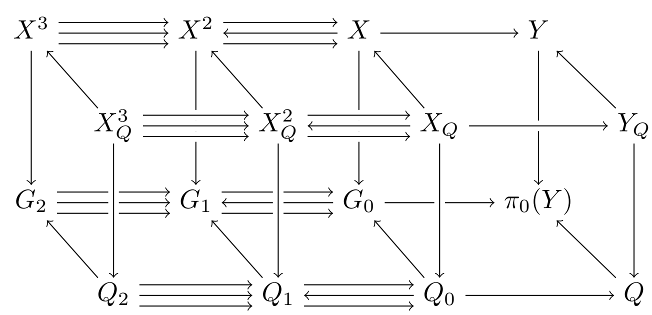
Proposition 6.13. *Suppose $f : X \to Y$ is a pre-Picard-Vessiot morphism of differential schemes with respect to $U : \mathscr{P} \Rightarrow \mathscr{S}$, where $\mathscr{S}$ is a class of morphisms stable under base change, and let $q : Q \to \pi_0(Y)$ be a morphism in $\mathscr{S}$. Then $q^*(f) = f_Q : X_Q \to Y_Q$ is pre-Picard-Vessiot and its Galois precategory is*

$$\mathrm{Gal}[q^*(f)] \simeq q^* \,\mathrm{Gal}[f] = \mathrm{Gal}[f] \times_{\pi_0(Y)} Q.$$

Moreover, if f is Picard-Vessiot for U, so is f_Q, and the same specialisation formula holds for Galois groupoids.

Proof. Assuming that f is pre-Picard-Vessiot, f_Q is an effective descent morphism for δ-$\mathscr{P}$ as a base change of an effective descent morphism f.

Writing $X^2 = X \times_Y X$ and $X^3 = X \times_Y X \times_Y X$, the Galois precategory of f is $\mathrm{Gal}[f] = (G_2, G_1, G_0)$ with $G_i = \pi_0(X^{i+1})$. By defining $Q_i = G_i \times_{\pi_0(Y)} Q = q^*(G_i)$, we obtain a diagram



where all the squares but those on the front and the rear face are cartesian. Since X^i is simple for $\mathscr{S}$, so is $X_Q^i \simeq (X_Q)^i$ and

$$\pi_0((X_Q)^{i+1}) = \pi_0(X^{i+1} \times_{G_i} Q_i) = \pi_0(X^{i+1} \times_{\pi_0(X^{i+1})} Q_i) \simeq Q_i,$$

whence we obtain that $\mathrm{Gal}[f_Q] = (Q_2, Q_1, Q_0)$, proving the specialisation formula.

If f is Picard-Vessiot, then the middle rear square is cartesian, i.e., $X \times_Y X \simeq X \times_{G_0} G_1$, so the middle front square is too, and we obtain

$$X_Q \times_{Y_Q} X_Q \simeq X_Q^2 \simeq X_Q \times_{Q_0} Q_1,$$

so f_Q is auto-split and thus Picard-Vessiot. □

7. Applications

7.1. **Quasi-projective Picard-Vessiot theory.**

qproj-gal-th

Theorem 7.2. *Let (K,δ) be a differential field of characteristic 0 with constants k, and write $S=\operatorname{Spec}(k)$. Let $f : X \to Y = \operatorname{Spec}(K,\delta)$ be a morphism of S-differential schemes such that*

(1) X is integral quasi-projective over Y and its only leaf is the generic point;
(2) f is auto-split, witnessed by a quasi-projective k-scheme G,

$$X \times_Y X \simeq X \times_{C(S)} C(G).$$

Then f is Picard-Vessiot, $\operatorname{Gal}[f]$ *is an S-algebraic group isomorphic to G, and there is an equivalence of categories*

$$\{\text{quasi-projective } S\text{-differential morphisms to } Y \text{ split by } f\}$$
$$\simeq \{\text{quasi-projective } S\text{-scheme actions of } G\}$$

Proof. We follow the template of 6.11 for $\mathscr{P}$ the fibred category of quasi-projective morphisms. By assumption (1) and 4.40, X is simple with categorical scheme of leaves S. By 4.51, f is of effective descent for δ-$\mathscr{P}$. Thus, X is Picard-Vessiot and 6.11 gives the desired equivalence. □

qproj-pv-corr

Corollary 7.3 (Quasi-projective differential Galois correspondence)**.** *With notation of 7.2, there is an order-reversing one-to-one correspondence between split S-differential quasi-projective fpqc quotients of $f : X \to Y$ in $\mathscr{A}$ and closed subgroups of the algebraic group $G = \operatorname{Gal}[f]$ which takes*

$$\begin{array}{ccc} & & X \\ & \swarrow & \big\downarrow f \\ P & & \\ & \searrow_{p} & \\ & & Y \end{array}$$

to

$$\operatorname{Gal}[X \to P].$$

Conversely, a closed subgroup G' corresponds to the quotient

$$X/G',$$

which is f-split by the quasi-projective scheme G/G'.

Moreover, this correspondence restricts to a one-to-one correspondence between split fpqc quotients P such that $P \to Y$ is Picard-Vessiot, and closed normal subgroups of G. In this case,

$$\operatorname{Gal}[P \to Y] \simeq \operatorname{Gal}[X \to Y]/\operatorname{Gal}[X \to P].$$

Proof. The claimed correspondence is more specific than a claim that could be extracted from 3.4, so we provide an explicit proof following an analogous strategy.

If $h : X \to P$ is f-split by $g : G \to Q$, and h is fpqc, then $f^*(h)$ is fpqc, hence an universal effective epimorphism by stacks-project [30, Tag 023P]. Since S_X preserves colimits as a left adjoint, it follows that $g = U(h) = S_X f^*(h)$ is a regular epimorphism, hence again an effective epimorphism in the presence of pullbacks. Since g corresponds to a quotient of an algebraic group, it is fpqc again by 5.9.

Conversely, if $G' \leq G$ is a closed subgroup, then, by 5.9, $g : G \to G/G' = Q$ is fpqc. The quotient $h : X \to P$ corresponding to it satisfies $f^*(h) \simeq C_X(g)$, whose underlying scheme morphis is fpqc since C_X acts as base change on g. Since f is fpqc, using the fact stacks-project [30, Tag 02YJ] that the properties of being 'faithfully flat and quasi-compact' are local in the fpqc topology, it follows that h is fpqc itself. □

Remark 7.4. The theory above shows that even in *linear Picard-Vessiot theory,* with $X = (A, \delta_A)$ the spectrum of a Picard-Vessiot ring over a differential field (K, δ), in order to extend the correspondence 5.10 to all closed subgroups od the Galois group, we are forced to consider split quasi-projective quotients of X.

Remark 7.5. The above theory applies to *strongly normal* differential Galois theory of kolchin-sn [19].

7.6. **Polarised quasi-projective differential Galois theory.**

polarised-pv-thm **Theorem 7.7.** *Let $f : (X, \delta_X) \to (Y, \delta_Y)$ be a morphism of S-differential schemes such that, with notation 4.43,*

(1) the underlying S-scheme morphism $X \to Y$ is fpqc;
(2) (X, δ_X) is simple for $\mathscr{S}$ with scheme of leaves G_0;
(3) there is an S-morphism $G_1 \to G_0$ such that

$$(X, \delta_X) \times_{(Y,\delta_Y)} (X, \delta_X) \simeq (X, \delta_X) \times_{(G_0,0)} (G_1, 0).$$

Then f is Picard-Vessiot for $U : \mathscr{P} \Rightarrow \mathscr{S}$, $\mathrm{Gal}[f]$ is the groupoid $(G_1 \rightrightarrows G_0)$ and we have an equivalence between the category of quasi-projective polarised S-differential morphisms $(P, \mathscr{L}_P) \to Y$ split by f and the category of quasi-projective polarised actions $(Q, \mathscr{L}_Q) \to G_0$ of $\mathrm{Gal}[f]$.

Proof. Using 4.51, we obtain that f is of effective descent for δ-$\mathscr{P}$. By 4.47, the forgetful functor δ-U reflects $\mathscr{S}$-universal categorical schemes of leaves. Hence, f is indeed Picard-Vessiot for U and we can apply the template Theorem 6.11. □

s:elliptic

7.8. **A parametrised family of strongly normal extensions.** This example is inspired by Kolchin's example of Weierstrassian extensions of differential fields kolchin-sn [19, III.6], with more explicit calculations in kovacic-tams03, kovacic-tams06 [20, 21] where the authors consider the strongly normal extension associated to a vector field on an elliptic curve. We are able to treat *families* of elliptic curves over a base scheme as parametrised families of strongly normal extensions.

We consider an elliptic curve scheme $E \to S$ with a globally defined invariant differential $\omega_E \in \Omega^1_{E/S}$ (the reader may wish to think of the Weierstrass family of elliptic curves

$$E \to S = \operatorname{Spec} \mathbb{Q}[u, v, (4u^3 + 27v^2)^{-1}]$$

obtained by projectivising the naive equation $y^2 = x^3 + ux + v$).

Let $p : (Y, \delta_Y) \to (S, 0)$ be a universal Bardavid quotient, and let

$$X = Y \times_S E.$$

It sheaf of differentials is

$$\Omega_{X/S} \simeq \pi_1^* \Omega_{Y/S} \oplus \pi_2^* \Omega_{E/S} = \pi_1^* \Omega_{Y/S} \oplus \langle \omega_X \rangle,$$

for some ω_X.

We endow X with a differential scheme structure over (Y, δ_Y) by stipulating

$$\omega_X \mapsto \alpha \in \Gamma(X, \mathscr{O}_X^\times),$$

and we write

$$f : (X, \delta_X) \to (Y, \delta_Y)$$

for the corresponding morphism of S-differential schemes.

We claim that the morphism

$$\eta = p \circ f : (X, \delta_X) \to (S, 0)$$

is a universal Bardavid quotient.

(1) Since f is proper smooth surjective, it is faithfully flat and therefore universally submersive surjective, and p is such by assumption, hence η is (universally) submersive and surjective.

(2) For the trajectory condition, we need to show that $\eta(x) = \eta(x')$ implies $\mathrm{Traj}(x) = \mathrm{Traj}(x')$. Assuming $\eta(x) = \eta(x')$ implies that $p(f(x)) = p(f(x'))$, so by the orbit condition on p, we obtain that $\mathrm{Traj}(f(x)) = \mathrm{Traj}(f(x'))$. It suffices to show that the restriction $f^\delta : X^\delta \to Y^\delta$ of f to leaves is injective. Indeed, given a leaf $y \in Y^\delta$, $f^{-1}(y)$ is the underlying space of $X_y = X \times_Y \mathrm{Spec}(\kappa(y))$, which is an elliptic curve over a differential field $\kappa(y)$ endowed with a non-vanishing vector field associated with a logarithmic differential equation $l\delta(x) = \alpha(y)$, and [19, 20] prove that X_y is classically simple in the sense that its only leaf is its generic point, i.e., X_y^δ is a singleton.

(3) For the sheaf condition, using the fact that f is proper with geometrically connected fibres, we obtain that

$$\mathrm{Const}(\eta_* \mathscr{O}_X) = \mathrm{Const}(\mathfrak{p}_*(f_* \mathscr{O}_X)) \simeq \mathrm{Const}(p_* \mathscr{O}_Y) \simeq \mathscr{O}_S.$$

The universality of the quotient is automatic because any base change $S' \to S$ results in the same situation that we started with.

Thus, we have that $\pi_0(X) = \pi_0(Y) = S$, X is simple with respect to arbitrary scheme morphisms and we claim that f is self-split, i.e.,

$$X \times_Y X \simeq X \times_{(S,0)} (E, 0).$$

The underlying scheme isomorphism is

$$\psi : X \times_S E \simeq Y \times_S E \times_S E \xrightarrow{\mathrm{id} \times (\mu, \pi_1)} Y \times_S E \times_S E \simeq (Y \times_S E) \times_Y (Y \times_S E) \simeq X \times_Y X,$$

where $\mu : E \times_S E \to E$ is the group operation on E.

In order to show that ψ is a morphism of differential schemes, it suffices to verify that the diagram

$$\begin{array}{ccc} \psi^* \Omega_{X \times_Y X/S} & \longrightarrow & \Omega_{X \times_S E/S} \\ & \searrow & \downarrow \\ & & \mathscr{O}_{X \times_S E} \end{array}$$

commutes. Bearing in mind that both $\Omega_{X \times_Y X/S}$ and $\Omega_{X \times_S E/S}$ are isomorphic to

$$\Omega_{Y \times_S E \times_S E/S} \simeq \pi_1^* \Omega_{Y/S} \oplus \pi_2^* \Omega_{E/S} \oplus \pi_3^* \Omega_{E/S},$$

we name the generators as

$$\psi^* \Omega_{X \times_Y X/S} = \pi_1^* \Omega_{Y/S} \oplus \langle \omega_{X,1}, \omega_{X,2} \rangle, \qquad \Omega_{X \times_S E/S} = \pi_1^* \Omega_{Y/S} \oplus \langle \omega_X, \omega_E \rangle.$$

The horizontal arrow acts as identity on $\pi_1^*\Omega_{Y/S}$ and maps

$$\omega_{X,1} \mapsto \omega_X + \omega_E, \quad \omega_{X,2} \mapsto \omega_X,$$

while the arrows to $\mathscr{O}_{X\times_S E}$ map

$$\omega_{X,1} \mapsto c, \quad \omega_{X,2} \mapsto c, \quad \omega_X \mapsto c, \quad \omega_E \mapsto 0,$$

so the diagram commutes. These considerations follow formally from properties of invariant differentials and do not require explicit calculations in coordinates as in kolchin-sn,kovacic-tams03 [19, 20].

It follows that the Galois groupoid is

$$\mathrm{Gal}[f] = E \rightrightarrows S.$$

Note that 6.13 explains the variation in parameters from S. If $s \in S(L)$ for a field L, then

$$\mathrm{Gal}[f_s] \simeq \mathrm{Gal}[f] \times_S \mathrm{Spec}(L) \simeq (E \rightrightarrows S) \times_S \mathrm{Spec}(L) \simeq E_s \rightrightarrows \mathrm{Spec}(L) \simeq E_s,$$

considered as an algebraic group over L. Hence, our Galois groupoid specialises to the classical differential Galois groups of strongly normal extensions associated to logarithmic-differential equations on elliptic curves $f_s : X_s \to Y_s$ uniformly in parameter s.

s:airy

7.9. **Galois groupoid of the Airy equation.** On the example of the Airy equation

$$y'' = xy,$$

we show that Galois theory of linear differential equations can be done in a more canonical way through a Galois groupoid, rather than following the classical route of constructing a Picard-Vessiot extension in a non-canonical way in order to obtain a Galois group.

Let $S = \mathrm{Spec}(k)$ be the spectrum of a field of characteristic 0, let $Y = \mathrm{Spec}\, k[x]$ and $X = \mathrm{Spec}(A)$ with $A = k[x, u, \det(u)^{-1}]$, a variant of the 'full universal solution algebra' $\bar{A} = k(x)[u, \det(u)^{-1}]$ of the Airy equation. We make the projection

$$f : X = Y \times_S \mathrm{GL}_2 = \mathrm{Spec}(A) \to Y$$

into a morphism of differential S-schemes by endowing X with a vector field

$$\frac{\partial}{\partial x} + u_{21}\frac{\partial}{\partial u_{11}} + u_{22}\frac{\partial}{\partial u_{12}} + xu_{11}\frac{\partial}{\partial u_{21}} + xu_{12}\frac{\partial}{\partial u_{22}},$$

which also determines the differential structure on Y. In terms of differentials,

$$\Omega_{X/S} = f^*\Omega_{Y/S} \oplus \pi_2^*\Omega_{\mathrm{GL}_2/S} = \langle dx, \omega_{11}, \omega_{12}, \omega_{21}, \omega_{22}\rangle,$$

where $\omega = du \cdot u^{-1}$ is the invariant differential on GL_2, the S-differential scheme structure on X and Y is given by assigning

$$dx \mapsto 1, \quad \omega \mapsto \begin{pmatrix} 0 & 1 \\ x & 0 \end{pmatrix}.$$

We claim that the composite

$$\eta : X \xrightarrow{\pi_2} \mathrm{GL}_2 \xrightarrow{\det} \mathbb{G}_m$$

is a geometric quotient.

(1) Since η is fpqc, it is universally submersive surjective.

(2) Given that we are in the affine setting, the sheaf condition follows from the folklore fact that

$$\mathrm{Const}(A) = k[\det(u), \det(u)^{-1}],$$

where $A = k[x, u, \det(u)^{-1}]$, i.e., the 'only' algebraic relation that a fundamental set of solutions of the Airy equation satisfies is that its Wronskian is a constant.

(3) For the orbit condition, let $p \in X(L)$ be a point with values in a field L, corresponding to a homomorphism $\alpha : A \to L$. The orbit map a_p is associated with an L-homomorphism

$$A \otimes_k L = L[x, u, \det(u)^{-1}] \to L[[t]], \quad f \mapsto \sum_n \frac{\alpha(\delta^n f)}{n!} t^n.$$

An explicit calculation shows that

$$x \longmapsto \alpha(x) + t,$$
$$u \longmapsto \begin{pmatrix} a(\alpha(x)+t) & b(\alpha(x)+t) \\ a'(\alpha(x)+t) & b'(\alpha(x)+t) \end{pmatrix} \alpha(u),$$

where a and b are the fundamental set of solutions of the Airy equation in $L[[t]]$ (with respect to the variable t) with Wronskian 1. By the same reasoning as above, we obtain that the kernel of this map is $(\det(u) - \det(\alpha(u)))$. In other words, we have that

$$\eta(p) = \eta(p') \text{ if and only if } O(p) = O(p').$$

Next, we claim that

$$\mathrm{coker}(\eta_* \mathscr{O}_X \xrightarrow{\eta_* \delta_X} \eta_* \mathscr{O}_X)$$

is a flat module on $\mathbb{G}_m$. Since everything is affine, we need to show that $M = \mathrm{coker}(A \xrightarrow{\delta_A} A)$ is a flat $k[z, z^{-1}]$-module, via the morphism associated to η, $k[z, z^{-1}] \to A$, $z \mapsto \Delta = \det(u)$.

By faithfully flat descent via $k \to \bar{k}$, we may assume that k is algebraically closed.

Since $k[z, z^{-1}]$ is a 1-dimensional regular ring, it is enough to show that M is torsion-free. In this case, this means that, if $f(\Delta)m \in \mathrm{im}(\delta)$ for some $m \in A$, and $f \in k[z, z^{-1}]$, then already $m \in \mathrm{im}(\delta)$. More explicitly, if we have $f(\Delta)m = \delta(g)$, then there must exist an $h \in A$ such that $m = \delta(h)$.

By clearing denominators using the fact that Δ is a constant, we may assume that $f \in k[z]$. Since k is algebraically closed, the polynomial f factors into linear factors, hence, by an inductive argument, we reduce to the case $f(z) = z - \lambda$, for some $\lambda \in k$. The above condition reduces to showing that

$$\mathrm{Const}(A/(\Delta - \lambda)) = k.$$

By the above arguments, we know that $(\Delta - \lambda)$ is a maximal δ-ideal in A and $\bar{A}$, and that

$$A/(\Delta - \lambda) \hookrightarrow \bar{A}/(\Delta - \lambda),$$

where the latter is a Picard-Vessiot ring over $k(x)$ by construction magid-lect [25, 3.4, 4.29], so

$$k \subseteq \mathrm{Const}(A/(\Delta - \lambda)) \subseteq \mathrm{Const}(\bar{A}/(\Delta - \lambda)) = k,$$

as required.

Moreover, since η is fpqc, using 4.36, we obtain that X is simple with respect to arbitrary scheme morphisms with

$$\pi_0(X) = G_0 = \mathbb{G}_m.$$

We claim that $f : X \to Y$ is self-split, i.e., there is a natural quotient morphism

$$\eta_1 : X \times_Y X \simeq Y \times_S \mathrm{GL}_2 \times_S \mathrm{GL}_2 \to G_1 = (\mathrm{GL}_2 \times_S \mathrm{GL}_2)/\,\mathrm{SL}_2\,.$$

that yields an isomorphism of differential schemes

$$X \times_Y X \simeq X \times_{(G_0,0)} (G_1, 0).$$

Indeed, for the last isomorphism in the definition of η_1, given a pair of invertible matrices (u, v), consider the augmented matrix

$$\begin{pmatrix} u_{11} & u_{12} & v_{11} & v_{12} \\ u_{21} & u_{22} & v_{21} & v_{22} \end{pmatrix}$$

and send it to the the homogeneous sextuple of determinants p_{ij} of its 2×2 minors for $1 \le i < j \le 4$, considered as the Plücker coordinates of the Grassmanian $\mathrm{Gr}[2, 4]$, satisfying the familiar relation $p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = 0$. The definition is SL_2 invariant, so it factors through $(\mathrm{GL}_2 \times_S \mathrm{GL}_2)/\,\mathrm{SL}_2$. Using the derivation on $X \times_Y X$, we explicitly verify that all the p_{ij} are constants, so η_1 is a differential scheme morphism to $(\mathrm{Gr}[2, 4], 0)$.

The torsor isomorphism

$$\begin{aligned} \psi : X \times_S (\mathrm{GL}_2 \times_S \mathrm{GL}_2)/\,\mathrm{SL}_2 &\simeq Y \times_S \mathrm{GL}_2 \times_S (\mathrm{GL}_2 \times_S \mathrm{GL}_2)/\,\mathrm{SL}_2 \\ &\longrightarrow Y \times_S \mathrm{GL}_2 \times_S \mathrm{GL}_2 \simeq X \times_Y X, \end{aligned}$$

takes a tuple $(y, u, [u_1, v_1])$ and maps it to (y, u, sv_1), where $s \in \mathrm{SL}_2$ is the unique matrix such that $u = su_1$. Its inverse is given as $(\pi_{Y\times_S \mathrm{GL}_2)}, \eta_1)$, i.e., in coordinates, by $(y, u, v) \mapsto (y, u, [u, v])$.

In order to show that ψ is a morphism of differential schemes, it suffices to verify that the diagram

$$\begin{array}{ccc} \psi^*\Omega_{X\times_Y X/S} & \longrightarrow & \Omega_{X\times_{G_0} G_1/S} \\ & \searrow & \downarrow \\ & & \mathscr{O}_{X\times_{G_0} G_1} \end{array}$$

commutes. Using the fact that both $\Omega_{X\times_Y X/S}$ and $\Omega_{X\times_{G_0} G_1/S}$ are isomorphic to

$$\Omega_{Y\times_S \mathrm{GL}_2\times_S \mathrm{GL}_2/S} \simeq \pi_1^*\Omega_{Y/S} \oplus \pi_2^*\Omega_{\mathrm{GL}_2/S} \oplus \pi_3^*\Omega_{GL_2/S},$$

we can name the generators, thought of as pullbacks of the invariant differentials on GL_2, as

$$\psi^*\Omega_{X\times_Y X/S} = \langle dx, \omega_1, \omega_2\rangle, \qquad \Omega_{X\times_{G_0} G_1/S} = \langle dx, \omega, \omega_0\rangle,$$

so that the horizontal arrow maps

$$dx \mapsto dx, \quad \omega_1 \mapsto \omega, \quad \omega_2 \mapsto \omega + \omega_0,$$

while the arrows to $\mathscr{O}_{X\times_S E}$ map

$$\omega_1 \mapsto \begin{pmatrix} 0 & 1 \\ x & 0 \end{pmatrix}, \quad \omega_2 \mapsto \begin{pmatrix} 0 & 1 \\ x & 0 \end{pmatrix}, \quad \omega \mapsto \begin{pmatrix} 0 & 1 \\ x & 0 \end{pmatrix}, \quad \omega_0 \mapsto 0,$$

so the diagram commutes.

From the self-splitting, we conclude that

$$\pi_0(X \times_Y X) \simeq G_1,$$

and our Galois theory gives the Galois groupoid

$$\begin{aligned}\mathrm{Gal}[f] &= G_1 \rightrightarrows G_0 \\ &= (\mathrm{GL}_2 \times_S \mathrm{GL}_2)/\,\mathrm{SL}_2 \rightrightarrows \mathrm{GL}_2\,/\,\mathrm{SL}_2 = \mathbb{G}_m.\end{aligned}$$

Intuitively, at least in the case where k is algebraically closed, the points of the object of objects G_0 correspond to a choice of a Picard-Vessiot extension with $\Delta = \lambda$, and the object of morphisms G_1 encodes the isomorphisms between different choices of λ. The stabiliser of a chosen object is its usual Picard-Vessiot Galois group SL_2.

The Malgrange groupoid in this example is given by casale-blazquez [7, 2.8] as

$$\mathbb{A}^1 \times_S \mathbb{A}^1 \times_S \mathrm{SL}_2 \rightrightarrows \mathbb{A}^1.$$

While both groupoids show that the symmetries of the Airy equation are governed by the group SL_2, they appear to do so in different ways.

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